Lecture 4-21: Basic topology of the real numbers

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Following part of Chapter 3 of the text, I present some basic definitions arising in something called the topology of the real line.

Recall that in the lecture on April 11 I defined a subset S of $\mathbb{R}$ to be closed if it contains the limit of any convergent series of its points. I warned you at that time that while there is also a notion of open set, an open set is *not* just one that is not closed! Now the time has come to define this notion.

Definition 3.2.1, p. 88

A subset S of $\mathbb{R}$ is called *open* if whenever $x \in S$ there is $\epsilon > 0$ (depending on x) such that the open interval $(x-\epsilon, x+\epsilon) = \{y \in \mathbb{R} : x-\epsilon < y < x+\epsilon\}$ is a subset of S. By convention, the empty set is also considered open.

In particular, an open interval (a, b) (or $a, \infty)$) is an open subset of $\mathbb{R}$, just as a closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (or $[a, \infty)$) is a closed subset. A half-open interval $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ is neither closed nor open.

The most fundamental property of open sets is then

Theorem 3.2.3, p. 89

Any union of open sets is open. Any finite intersection $\cap_{i=1}^{m} U_i$ of open sets $U_1, \ldots, U_m$ is open.

The assertion about unions is clear; the one about intersections follows since if $x \in U_i$ for all i, so that there are $\epsilon_1, \ldots, \epsilon_m$ with 9$x - \epsilon_i, x + \epsilon_i) \subset U_i$, then we have $(x - \epsilon, x + \epsilon) \subset \cap_{i=1}^{m} U_i$, where ϵ is the minimum of $\epsilon_1, \ldots, \epsilon_m$.

The relationship between open and closed sets is given by

Theorem 3.2.13, p. 92

A set $U \in \mathbb{R}$ is open if and only if its complement C (consisting of all $x \in \mathbb{R}$ not in U) is closed.

Indeed, if U is open and (s_n) is a convergent sequence of points not in U with limit s, then we cannot have $s \in U$, lest there be an $\epsilon > 0$ with $(s - \epsilon, s + \epsilon) \subset U$, contradicting $|s_n - s| < \epsilon$ for all indices n larger than a fixed one N. If instead U is closed and $x \notin U$, then suppose for every i there is $x_i \in (x - 1/i, x + 1/i), x \in U$. Then the sequence (x_n) clearly converges to $x \notin U$, a contradiction, since the x_i lie in U. Hence there must be an i with $(x - 1/i, x + 1/u) \subset C$ and C is open.

Since the complement of a union of sets is the intersection of their complements, we get

Theorem 3.2.14, p. 92

Any intersection of closed sets is closed. Any finite union $\cup_{i=1}^{m} C_i$ of closets C_i is closed.

In particular, given any subset S of $\mathbb{R}$, the intersection of all closed subsets of $\mathbb{R}$ containing S is the unique smallest closet of $\mathbb{R}$ containing S, called its closure and denoted $\overline{S}$. See Definition 3.2.11 on p. 91.

Thus, as a well-known saying goes, "sets are not like doors": most subsets of $\mathbb{R}$ are neither open nor closed. One property of doors does however (almost) carry over to sets.

Theorem; cf. Theorem 3.4.6, p. 104

The only subsets of $\mathbb{R}$ that are both open and closed are the empty set $\emptyset$ and $\mathbb{R}$ itself.

Proof.

Suppose for a contradiction that A is a nonempty proper open and closed subset of $\mathbb{R}$ with complement B. Then there is an interval $I = [a, b]$ such that both A and B have nonempty intersection with I. For definiteness assume $b \in B$. Since B is open there is $\epsilon > 0$ with $(b - \epsilon, b + \epsilon) \subset B$, so that the supremum c of $A \cap I$ is strictly less than b; similarly it is strictly greater than a. Then there is a sequence (x_n) of points in A with $x_n \in (c - 1/n, c]$ and likewise a sequence (y_n) of points in B with $y_i \in [c, c + 1/n)$. The sequences $(x_n), (y_n)$ then both converge to c, forcing $c \in A \cap B$, a contradiction. Note that if we define a subset of an interval I (open, half-open, or closed) to be open (or closed) in I if it is the intersection of an open (or closed) subset of $\mathbb{R}$ and I, then the same argument shows that the only open and closed subsets of I are the empty set and I itself. $\square$

Example

Next I consider a very interesting example of a closed subset of $\mathbb{R}$ very different from a closed interval. This is the Cantor set C, discussed pn pp. 85-88 in the text. It is easiest to define C as consisting of all points $\sum_{i=0}^{\infty} a_i 3^{-i}$, where coefficient a_i is 0 or 2. Note that two such expansions $\sum a_i 3^{-i}, \sum b_i 3^{-i}$ are equal if and only if $a_i = b_i$ for all i. Given a sum $x = \sum_{i=0}^{\infty} a_i e^{-i}$ in C, let $f(x) = \sum_{i=0}^{\infty} b_i 2^{-i}$, where $b_i = a_i/2 = 0$ if $a_i = 0$ and $b_i = 1$ if $a_i = 2$. This map is surjective (but not injective), whence C is uncountable (since the range of f is the entire interval [0, 1], which is uncountable).

This last observation suggests that C is large; but on the other hand another way of describing C makes it seem small. Start with the closed interval $C_1 = [0, 1]$ and remove the open middle third $(1/3, 2/3)$, obtaining the union C_2 of the two disjoint closed intervals $I_1 = [0, 1/3]$ and $I_2 = [2/3, 1]$, of total length $2/3$. Then remove the open middle third of each I_i, obtaining thereby a disjoint union C_3 of four closed intervals of total length $4/9$. Inductively, if C_i has been defined and is the union of 2^i disjoint closed intervals $[a_i, b_i]$ each of length 3^{-i}, so that the total length of C_i is $(2/3)^i$, then remove the open middle third $(\frac{2a+b}{3}, \frac{a+2b}{3})$ from this interval, thereby producing a union C_{i+1} of 2^{i+1} disjoint closed intervals of total length $(2/3)^{i+1}$.

Then the Cantor set C can alternatively be realized as the intersection of all the C_i. Since each C_i is closed, so is C. Since the length of C_i (that is, the total length of the closed intervals making up C_i) is $(2/3)^i$ and $(2/3)^i \to 0$ as $i \to \infty$, one can argue that the length of C is 0, so that C is small. The two contradictory indications given above of the size of C are what makes this set interesting.

A subset S of $\mathbb{R}$ that is *not* the disjoint union of its nonempty intersections $S \cap U, S \cap C$ with and open set U and closed set C is called connected; see Definition 3.4.4 on p. 104. The argument above shows that any interval in $\mathbb{R}$ (open, half-open, or closed, and bounded or unbounded on either end) is connected. On the other hand, no other subset S of $\mathbb{R}$ is connected, for if a subset S contains points a, b but not c, with $a < c < b$, then it is the disjoint union of its intersections with $(-\infty, c)$ and (c, ∞), which are the same as its respective intersections with $(\infty, c]$ and $[c, \infty)$, so that each of these intersections is nonempty, open, and closed in S.

There is another very important kind of subset of $\mathbb{R}$, called compact. A compact subset of $\mathbb{R}$ is one that is both closed and bounded; equivalently, one such that every sequence of points in it has a convergent subsequence whose limit also lies in it (see Definition 3.3.1 on p. 96). Thus a closed interval $[a, b]$ is compact but an open interval (a, b) (with $a < b$) is not. Compact and connected subsets of $\mathbb{R}$ play a very important role in two theorems you will see later in the course, called the Intermediate Value Theorem and Extreme Value Theorem. I will discuss these theorems when I start covering continuous functions after the midterm.