Lecture 4-2: The real numbers, continued

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I will begin by sketching the construction of the real numbers, as given in section 8.6 of the text. This will give a very clear indication of how one uses the set of rational numbers itself to plug up its own holes. The basic idea is to construct any real number r as the set of C_r of rational numbers q with $q < r$, being careful *not* to consider at the same time the set of rational numbers q with $q \leq r$, as then these two possibly different sets of rational numbers would then have to denote the same real number.

More precisely, I call a set S of rational numbers a cut if it is nonempty, bounded above, has no greatest element, and contains any rational number $x < y$ whenever it contains y (see p. 298). Then the real numbers (by definition) are exactly the cuts. Given cuts C_x, C_y defining the respective numbers x, y, the condition for x to be less than or equal to y is clearly the set-theoretic condition that $C_x \subseteq C_y$. Then the least upper bound of a nonempty set $\{C_i : i \in I\}$ of cuts that is bounded above (so that there is $r \in \mathbb{Q}$ with $r \notin C_i$ for any i) is just the union C of all the C_i, which clearly satisfies the definition of cut. This very simple definition thus yields the Least Upper Bound Property as a consequence.

Next I have to define the arithmetic operations on these cuts so as to satisfy the axioms of a field. For addition this is straightforward: given two cuts C, D, define their sum $C + D$ to consist of all sums $c + d$ as c runs over C and d runs over D. Taking negatives is already a little tricky; since $x < y$ if and only if $-y < -x$, we define the negative $-C$ of a cut C by first taking all $-d$ as d runs over the rational numbers *not* in C, and then removing the largest element if there is one (see pp. 298-99). Thus the cut defining -1 consists of all rational numbers r with $r \leq -1$, with -1 removed, so that in the end it consists exactly of the rationals r with $r < -1$. By contrast, the cut defining $-\sqrt{2}$ consists exactly of the negatives $-r$ of all rational numbers $r > \sqrt{2}$; since $-\sqrt{2}$ is not rational, this set has no largest number, so no number needs to be removed from it to make it into a cut. The difference $C - D$ of two cuts C, D is just the sum $C + (-D)$.

Multiplication is even trickier: the problem is that it is *not* true that if $x < y$ and $z < w$, then $xy < zw$, though this *is* true if x, y, z, w are all positive. We there for start by defining a cut C to be *positive* if $0 \in C$. Then the product CD of two positive cuts C, D consists of all product cd of positive rational numbers c, d, lying in C, D, respectively, together with all rational numbers $r \leq 0$. We multiply negative cuts via the rules
$(-C)D = C(-D) = -(CD), (-C)(-D) = CD$. The multiplicative inverse C^{-1} of a positive cut C then consists of all d^{-1} as d runs over the rational numbers not in C, with the largest number removed if it has one, together with all rational $e \leq 0$. We extend multiplicative inverses to negative cuts by decreeing that $(-C)^{-1} = -C^{-1}$. If $C_0 = \{r \in \mathbb{Q} : r < 0\}$ is the cut defining the real number 0, then C_0^{-1} is not defined. Then one can check that the set of real numbers satisfies all the properties of an ordered field. See pp. 301-3.

The least upper bound property immediately implies that

The Archimedean Property; see Theorem 1.4.2, p. 21

The set $\mathbb{N}$ of positive integers is not bounded above; equivalently, given $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ with $x < n$.

Proof.

Indeed, if $\mathbb{N}$ were bounded above, then it would have a least upper bound x, whence $x - 1$ is not an upper bound for $\mathbb{N}$ and there is $n \in \mathbb{N}$ with $n > x - 1$. But then $n + 1 \in \mathbb{N}$ and $n + 1 > x$, a contradiction. $\square$

Note that an equivalent formulation of this property states that given any positive real numbers a, b we have $na > b$ for some $n \in \mathbb{N}$; to see this just choose $n \in \mathbb{N}$ with $n > \frac{b}{a}$.

An important consequence of the construction of the real numbers from the rational numbers is

Theorem 1.4.3, p. 22

For any real numbers x, y with $x < y$ there is a rational number z with $x < z < y$.

Proof.

The proof is quite easy, given the construction of the real numbers. Indeed, if $x < y$, then since y is the least upper bound of the set C_y, it follows that x is not an upper bound of this set, so that we can find a rational $z \in C_y$ with $x < z$. Then $x < z < y$, as desired. $\square$

This result is expressed by saying that the rational numbers are dense in $\mathbb{R}$.

An equivalent formulation of Theorem 1.4.3 says that given any $x \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$, there is $y \in \mathbb{Q}$ with $|x - y| < \epsilon$; in words, any real number can be approximated arbitrarily closely by rational numbers.

The corresponding property of $\mathbb{Z}$ within $\mathbb{R}$ is

Proposition

For any $c \in \mathbb{R}$ there is exactly one integer k in the half-open interval $[c, c+1)$.

I omit the straightforward proof.