Lecture 4-18: Rearrangements of series and double summations

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After digressing briefly to give a remarkable example of a series which converges by Dirichlet's Test, I give a precise account of rearrangements of series, showing that they do not affect the sums of absolutely convergent series but can drastically alter the sums of conditionally convergent ones.

First a rather strange-looking digression. By standard trig identities $\cos\frac{2i+1}{2}x = \cos(i+\frac{1}{2})x = \cos ix \cos\frac{x}{2} - \sin ix \sin\frac{x}{2}$, $\cos(i-\frac{1}{2})x =$ $\cos ix \cos\frac{x}{2} + \sin ix \sin\frac{x}{2}$, whence $cos(i-\frac{1}{2})x - \cos(i+\frac{1}{2})x = 2\sin ix \sin\frac{1}{2}x$. Adding for $1 \le i \le n$, we get $\sum_{i=1}^{n} \sin ix = \frac{\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}}{2\sin\frac{x}{2}}$ for any x that is not a multiple of 2π.

It follows that for any fixed x that is not a multiple of 2π, the partial sums of the series $\sum_{i=1}^{\infty} \sin ix$ are bounded; the same is true trivially if x is a multiple of 2π, since then all terms of the series are 0. By Dirichlet's test, any series $\sum_{i=1}^{\infty} \sin ixy_i$ with $y_1 \geq y_2 \geq \ldots, y_i \to 0$, converges; in particular, $\sum_{i=1}^{\infty} \frac{\sin ix}{i}$ converges. Such a series is (a special case of) a Fourier series; such series play a prominent role in physics, engineering, and mathematics. Although series $\sum_{i=1}^{\infty} \sin ixy_i$ as above converge, they typically do so conditionally and thus very slowly, so that it is difficult even to estimate their sums with a computer.

I now formally define rearrangements of series.

Definition 2.7.9, p. 75

A series $\sum_{k=1}^{\infty} b_k$ is a rearrangement of a series $\sum_{k=1}^{\infty} a_k$ if there is a bijection f from $\mathbb{N}$ to itself such that $b_k = a_{f(k)}$ for all k.

Then we have

Theorem 2.7.10, p. 75

Any rearrangement of an absolutely convergent series $\sum_{k=1}^{\infty} a_k$ converges to the same sum.

Proof.

Suppose $\sum a_k$ converges absolutely and $\sum a_k = A$. Given $\epsilon > 0$ there is then an index N such that $\sum_{n=N}^{\infty} |a_n|$ converges to a number less than $\epsilon/2$. Given ϵ and a rearrangement $\sum_{k=1}^{\infty} b_k$ of $\sum_{k=1}^{\infty} a_k$ choose M large enough that all terms $a_1, \ldots, a_N$ appear among $b_1, \ldots, b_M$. Then any partial sum $\sum_{k=1}^{n} b_k$ for $n > M$ differs from $s_N = a_1 + \ldots a_N$ by some terms a_i with $i > N$, the sum of the absolute values of these terms being less than $\epsilon/2$; in turn s_N differs from A by less than $\epsilon/2$. Thus the partial sums of $\sum b_k$ converge to A as well. $\square$

Example

On the other hand, consider the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k-1}/k$, or more generally any series S that can be written as the difference $\sum p_k - \sum q_k$ of subseries with nonnegative terms such that $p_1 \geq p_2 \geq \ldots, p_i \to 0$ and similarly for the q_i. Suppose further that both series $\sum p_k, \sum q_k$ diverge. Then I claim that there is a rearrangement of S converging to any desired real number, say 1000. Indeed, start with the terms p_k. Since $\sum p_k$ diverges, there is a least index N with $\sum_{k=1}^{N} p_k > 1000$. Then $\sum_{k=1}^{N} p_k$ lies between $\sum_{k=1}^{N-1}$, which is less than 1000, and $1000 + p_N$. Henceforth all partial sums larger than 1000 will be less than $1000 + p_N$. Next, since $\sum q_k$ diverges, there is a least M such that $S_M = \sum_{k=1}^{N} p_k - \sum_{k=1}^{M} q_k < 1000$, and then $S_M > 1000 - q_M$. Next, the sum $\sum_{k=N+1}^{\infty} p_k$ still diverges, so there is a least $N_1 > N$ with $S_{N_1} = S_M + \sum_{k=N+1}^{N_1} p_k > 1000$; we have $S_{N_1} < 1000 + p_{N_1}$. Henceforth all partial sums less than 1000 will be larger than $1000 - p_M$.

Example

Continuing in this way and using that $p_n, q_n \to 0$ as $n \to \infty$, we get a rearrangement of S that converges to 1000. Given any real numbers A, B with $A > B$, we can also rearrange S so that the limit superior of the partial sums of the rearranged series is A while the limit inferior of these partial sums is B.

The moral of this example is that absolutely convergent series are much better behaved than conditionally convergent ones and one has much more freedom to manipulate them algebraically. In particular, given a double infinite summation $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$, we can group the terms in various ways to rewrite this double series as a single one. For example, if $\sum_{i=1}^{\infty} a_{ij}$ converges for each fixed j, say to s_j, and if $\sum_j s_j$ converges, then we can define $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ to be $\sum_{j=1}^{\infty} s_j$. Similarly, if instead $\sum_{j=1}^{\infty} a_{ij}$ converges for each fixed i, say to t_i, and if $\sum_i t_i$ converges, then we can define $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ to be $\sum_{i=1}^{\infty} t_i$.

In general, however, if both of these definitions apply to $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$, there is no reason to expect them to give the same value. There is an example at the beginning of Chapter 2 in the text . It turns out that an especially convenient way to group the terms is by diagonals, so that given the double sum above, we set $c_n = \sum_{i=1}^{n} a_i b_{n-i}$ and define $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ to be $\sum_{n=2}^{\infty} c_n$ if this sum converges. In particular, given two infinite series $\sum_{i=1}^{\infty} a_i, \sum_{j=1}^{\infty} b_j$, we define their Cauchy product to be $\sum_{k=2}^{\infty} c_k$, with the c_k defined as above (see pp. 82,84). The motivation for making this definition comes from power series; that is, (families of) series $\sum_{k=0}^{\infty} a_k x^k$, one for every $x \in \mathbb{R}$. Collecting coefficients of each fixed power of x, we see that the product $\sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j$ is $\sum_{k=0}^{\infty} c_k x^k$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$.

In general, provided that a series $S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges absolutely in the sense that $\sum_{k=2}^{\infty} d_k$ converges, where $d_k = \sum_{i=1}^{k} |a_i b_{k-i}|$, then any arrangement of the terms of S converges and any two arrangements of these terms converge to the same value. Moreover one has

Merten's Theorem

Suppose that $\sum_{n=0}^{\infty} a_n$ converges absolutely with sum A and $\sum_{n=0}^{\infty} b_n$ converges, not necessarily absolutely, to B. Then $\sum_{n=0}^{\infty} c_n = AB$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Proof.

First suppose that $B = 0$. I must show that $C_n \to 0$ as $n \to \infty$, where $C_n = \sum_{k=0}^{n} c_k$. Let $B_n = \sum_{k=0}^{n} b_k, \alpha = \sum_{k=0}^{\infty} |a_k| < \infty$. I then have $C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$ for all n and $B_n \to 0$ as $n \to \infty$, whence given $\epsilon > 0$ there is N such that $|C_n| \leq |a_n B_0 + \cdots + a_{n-N} B_N| + \epsilon\alpha$ for $n > N$. The $|B_i|$ are bounded above, say by M; then this last sum is bounded by $M(N+1)\epsilon + \epsilon\alpha$ if n is large enough. Hence $C_n \to 0$ as $n \to \infty$, as desired. In general, set $b_0' = b_0 - B, b_n' = b_n$ for $n > 0$ and apply the previous argument to the a_i and b_i'. $\square$

In general we do not have $C = AB$ if we merely assume that $\sum a_n$ and $\sum b_n$ converge respectively to A and B; for example, if $a_k = b_k = \frac{(-1)^k}{\sqrt{k}}$, then $\sum a_k, \sum b_k$ both converge by the Alternating Series Test, but $|c_{2n}| \geq \frac{n+1}{2n}$, so $\sum c_n$ does not converge.