Lecture 4-16: Convergence tests for infinite series

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Having covered sequences in a fair amount of detail in previous lectures the time has come to study the most important examples, namely infinite series, more deeply. I presented the Alternating Series Test for convergence of a series with positive and negative terms last time; today I will begin with series with nonnegative terms, but then look at arbitrary series.

I begin by first noting that if the series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ converge to S and T, respectively, then $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n - b_n)$ converge to $S + T$ and $S - T$, respectively. This follows at once from the definition of infinite series and the limit laws for addition and subtraction. Similarly, $\sum_{n=1}^{\infty} ca_n$ converges to cS. Next observe that if $\sum_{i=1}^{\infty} a_i$ converges, then $a_n \to 0$ as $n \to \infty$. This follows since we have $a_n = s_n - s_{n-1}$, where $s_n = \sum_{i=1}^{n} a_i$ is the nth partial sum, so that if s_n converges to S, then so does s_{n-1} and the difference $s_n - s_{n-1}$ converges to 0. The harmonic series $\sum_{i=1}^{\infty} 1/i$ shows that the converse of this last result is false: we have $1/n \to 0$, but $\sum_{i=1}^{\infty} 1/i$ diverges.

You have seen that an infinite series $\sum_{i=1}^{\infty} a_i$ with $a_i \geq 0$ for all i (or even for all i larger than some index N) converges if and only if its partial sums are bounded. The simplest way to verify this last condition is to compare the partial sums to those of a known series.

Comparison Test: Theorem 2.7.4, p. 73

If $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ are two series with $0 \leq a_k \leq b_k$ for all k, then if $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$; likewise, if $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

Proof.

This follows since any partial sum $s_n = \sum_{k=1}^n a_k$ of the a_k is at most the corresponding partial sum $t_n = \sum_{k=1}^n b_k$ of the b_k, so that if the t_n are bounded, so are the s_n, and if conversely the s_n are unbounded, then so too are the t_n. $\square$

The applicability of this result is vastly extended by the following one.

Limit Comparison Test

Suppose that $\sum a_k, \sum b_k$ are two series with $a_k, b_k \geq 0$ for all k. Assume that $L = \lim_{k\to\infty}(a_k/b_k)$ exists and is finite and nonzero. Then $\sum a_k$ converges if and only if $\sum b_k$ does.

Proof.

Choosing $\epsilon = L/2$, we find that there is an index N such that $\frac{L}{2}b_n \leq a_n \leq \frac{3L}{2}b_n$ for $n > N$, whence a typical partial sum $s_n = \sum_{k=N}^{n} a_k$ is bounded between $\frac{L}{2}\sum_{k=N}^{n} b_k$ and $\frac{3L}{2}\sum_{k=n}^{N} b_k$, for any $n > N$. Thus $\sum_{k=N}^{\infty} b_k$ converges if an only if $\sum_{k=N}^{\infty} a_k$ does, or if and only if $\sum b_k$ converges, or if and only if $\sum a_k$ does, as desired. $\square$

Thus for example we know that $S = \sum_{k=1}^{\infty} \frac{3k-7}{k^2-2k+4}$ diverges by limit comparison with the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$, since $\lim_{k\to\infty} \frac{\frac{3k-7}{k^2-2k+4}}{\frac{1}{k}} = 3$ is finite. Here I am using the facts that $ck^m \to 0$ as $k \to \infty$ for any constant c and strictly negative integer m. Note that while not all terms of the series S are nonnegative, all but finitely many are; this is enough to apply the test.

Probably the single most frequently applied convergence test is the following one.

Ratio Test: Exercise 2.7.9, p. 78

Given a series $\sum_{k=1}^{\infty}$ with $a_k \geq 0$ for all but finitely many k, suppose that $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ exists. Then $\sum a_k$ converges if $L < 1$ and diverges if $L > 1$. No information can be deduced if $L = 1$.

Proof.

If $L < 1$, then choose M with $L < M < 1$. We then have $a_{n+1} \leq Ma_n$ for $n > N$, say, whence by induction $0 \leq a_n \leq M^{n-N}a_N$ for $n > N$. Then $\sum_{n=N}^{\infty} a_n$ converges by comparison with the geometric series $\sum_{n=N}^{\infty} a_N M^{n-N}$, whence so too does the full series $\sum_{k=1}^{\infty} a_k$. If $L > 1$, then choose M with $L > M > 1$. We have $a_{n+1} \geq Ma_n$ for $n > N$ for some N, whence by induction $a_n \geq a_N > 0$ for all $n \geq N$, implying that $a_n \not\to 0$ as $n \to \infty$. Then $\sum a_n$ cannot converge. To illustrate the last assertion, consider the two series $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$. These are p-series with $p = 1$ in the first case and $p = 2$ in the second; the first diverges but the second converges. $\square$

Henceforth I will be deliberately vague about the initial values of my indices of summation. The results apply for any initial values. To apply convergence tests to series $\sum a_k$ for which not all terms are nonnegative, we make the following definition.

Definition 2.7.8, p. 74

A series $\sum a_k$ converges *absolutely* if $\sum |a_k|$ converges. if $\sum a_k$ converges but not absolutely, then it is said to converge *conditionally*.

Then we have

Theorem 2.7.6, p 73

If $\sum a_k$ converges absolutely, then $\sum a_k$ converges.

Indeed, if $\sum a_k$ converges absolutely and $b_k = a_k + |a_k|$, then $0 \leq b_k \leq 2|a_k|$ for all k, whence $\sum b_k$ converges by comparison with $\sum 2|a_k|$; but then so does $\sum a_k$, since $a_k = b_k - |a_k|$ for all k.

I turn now to series of the form $\sum x_k y_k$; note that any alternating series takes this form, with $x_k = (-1)^k$. I will start with a simple formula, analogous to integration by parts, but applied to finite sums rather than integrals.

Summation by parts: Exercise, 2.7.12, p. 78

Let (x_n) and (y_n) be sequences and set $s_m = x_1 + \ldots + x_m$ for $m \leq n, s_0 = 0$. Then $\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$.

This follows at once by setting $x_j = s_j - s_{j-1}$ and collecting coefficients of each s_j. The beautiful consequence for series is then

Dirichlet's Test: Exercise 2.7.14, p. 79

Suppose that the partial sums of the series $S = \sum x_k$ are bounded, while the series $\sum y_k$ satisfies $y_1 \geq y_2 \geq \ldots, y_k \to 0$ as $k \to \infty$. Then S converges.

Proof.

I will show that the partial sums t_n of S form a Cauchy sequence. For $m < n$ the difference $t_n - t_m$ is $\sum_{k=m+1}^{n} x_k y_k$, which equals $s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1})$ by summation by parts, where s_n is the nth partial sum of the x_k. By hypothesis there is $B \in \mathbb{R}$ with all s_i appearing on the right side satisfying $|s_i| < B$. Since the sum $\sum_{j=m+1}^{\infty}(y_j - y_{j+1})$ telescopes and is equal to y_{m+1} and $y_n \to 0$ as $n \to \infty$, given $\epsilon > 0$, there is an index N with $y_n < \frac{\epsilon}{3B}$ for $n > N$, whence $|t_n - t_m| < \epsilon$ for $n, m > N$. Hence the sequence (t_n) is Cauchy and must converge. □

As a very special case we recover the Alternating Series Test, since given a series $\sum_{k=1}^{\infty}(-1)^{k-1}y_k$ satisfying the hypotheses of that test, the partial sums of $\sum_{k=1}^{\infty}(-1)^{k-1}$ are indeed bounded while $y_1 \geq y_2 \geq \ldots$ and $y_i \to 0$ as $i \to \infty$. But the Dirichlet test is much more general, allowing for very different patterns of signs than the alternating one. I will give a striking example next time.