Lecture 4-14: Cauchy sequences and limits superior and inferior

April 14, 2025

I will wrap up sequences with accounts of limits superior and inferior and of Cauchy sequences.

I showed last week that every bounded monotone sequence converges and that every sequence has a monotone subsequence. I now want to give a precise measure of how far a divergent sequence is from being convergent. To this end, I will attach two quantities to any sequence s_n, calling them its limits superior and inferior and denoting them by $\overline{\lim}\, s_n, \underline{\lim}\, s_n$, respectively.

If the sequence s_n is not bounded above, I write $\overline{\lim}\, s_n = \infty$; if s_n diverges to -∞, in the sense that given any $M \in \mathbb{R}$ there is an index N with $s_n < M$ whenever $n \geq N$, then I write $\overline{\lim}\, s_n = -\infty$. Otherwise, for each index n, the supremum M_n of the set $S_n = \{s_n, s_{n+1}, \ldots\}$ of the set of all terms s_i with $i \geq n$ is defined (and finite) and the sequence M_n of suprema is decreasing and bounded below, so that this sequence converges, say to M. I write $M = \overline{\lim}\, s_n$ and call M the limit superior of (s_n). Similarly, if s_n is not bounded below, I write $\underline{\lim}\, s_n = -\infty$, while if this sequence diverges to ∞ I write $\underline{\lim}\, s_n = \infty$. Otherwise the infimum m_n of the set S_n is defined for all n and the sequence m_n of infima is increasing and bounded above, so has a limit equal (by definition) to $\underline{\lim}\, s_n$. This limit is called the limit inferior of (s_n). See p. 60 of the text.

If $\overline{\lim} s_n = L < \infty$ then we can choose an index n_1 such that $L - 1 < s_{n_1} < L + 1$ and inductively an index $n_{k+1} > n_k$ such that $L - \frac{1}{k+1} < s_{n_{k+1}} < L + \frac{1}{k+1}$; then the subsequence $t_k = s_{n_k}$ converges to L. Thus the limit superior of a sequence is the the largest possible limit of a subsequence, allowing for this purpose $\pm\infty$ as limits; similarly the limit inferior of a sequence is the smallest possible limit of a subsequence, again allowing $\pm\infty$ as limits. The sequence s_n converges to L if and only if $\overline{\lim}\, s_n = \underline{\lim}\, s_n = L$. The difference between $\underline{\lim}\, s_n$ and $\overline{\lim}\, s_n$ provides the promised measure of divergence of s_n whenever this sequence diverges.

Next I define the notion of Cauchy sequence. The idea is to produce a criterion for deciding when a sequence converges without knowing its limit (for example, you already know that a bounded monotone sequence must converge). The intuition here is fairly simple: if the terms of a sequence s_n are getting close to a limit L, then they must be getting close to each other. Accordingly I define s_n to be Cauchy if for every $\epsilon > 0$ there is an index N such that whenever $n, m \geq N$ one has $|s_n - s_m| < \epsilon$ (see Definition 2.6.1, p. 66). Then I have

Theorem 2.6.4, p. 67

The sequence s_n converges if and only if it is Cauchy.

Proof.

First suppose that s_n converges, say to L. Given ϵ choose an index N such that $|s_n - L| < \frac{\epsilon}{2}$ for $n \geq N$. For $n, m \geq N$ I then get $|s_n - s_m| \leq |s_n - L| + |L - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ by the Triangle Inequality. Next suppose that s_n is Cauchy and choose an index N such that $|s_n - s_m| < 1$ for $n, m \geq N$. In particular, $|s_n - s_N| < 1$ for $n \geq N$, whence $|s_n| \leq |s_N| + 1$ for $n \geq N$. It follows at once that s_n is bounded by the maximum of $|s_1|, \ldots, |s_{N-1}|, |s_N| + 1$ (as in the proof that every convergent sequence is bounded). Next let t_n be a monotone subsequence of s_n; write $t_k = s_{n_k}$ for its kth term. Then t_n is also bounded and so must converge, say to L. Now I claim that s_n also converges to L. □

Proof.

To prove this let $\epsilon > 0$. Then there is an index K such that $|t_n - L| < \epsilon/2$ for $n \geq K$. Choose M so that $|s_n - s_m| < \epsilon/2$ for $n, m \geq M$ and let $N = \max\{K, M\}$ Then for $m \geq N$ we have $n_m \geq m$ whence $|s_m - L| \leq |s_m - s_{n_m}| + |s_{n_m} - L| < (\epsilon/2) + (\epsilon/2) = \epsilon$, as desired. □

This theorem turns out to be equivalent to the Least Upper Bound Property.

One can use Cauchy sequences as an alternative to Dedekind cuts to give another construction of the real numbers. Define two Cauchy sequences a_n, b_n of *rational* numbers to be *equivalent* if the interleaved sequence $a_1, b_1, a_2, b_2, \ldots$ is Cauchy (where the nth term of this sequence is a_m if $n = 2m - 1$ is odd while the nth term is b_m if $n = 2m$ is even). Then I take a real number to be an equivalence class of Cauchy sequences of rational numbers. I add, subtract, and multiply two Cauchy sequences s_n, t_n term by term, taking the sum, difference, and product to be $s_n + t_n, s_n - t_n, s_n t_n$, respectively; these operations are well defined on equivalence classes.

Similarly the quotient of two Cauchy sequences s_n, t_n is taken to be (the equivalence class of) the sequence $\frac{s_n}{t_n}$, provided that $t_n = 0$ for only finitely many n; one can show that $\frac{s_n}{t_n}$ is Cauchy provided that t_n is not equivalent to the 0 sequence. A Cauchy sequence s_n is called positive if there is a positive rational number r with $s_n > r$ for all but finitely many n; one Cauchy sequence s_n is less than another one t_n if $t_n - s_n$ is positive.

So far the technical complications of defining the arithmetic operations on Cauchy sequences have been comparable to those of defining the same operations on Dedekind cuts. Things get quite messy, however, when one verifies that any Cauchy sequence of equivalence classes of Cauchy sequences converges. I will omit the details of this.

I will conclude by using the Monotone Convergence Theorem in a very nice way, by showing that certain series always converge (though typically very slowly). One has

Alternating Series Test, Theorem 2.7.7, p. 74

Given a series $\sum k = 1^{\infty}(-1)^{k+1}a_k$, assume that

- $a_k \geq 0$ for all k;
- $a_k \geq a_{k+1}$ for all k; and
- $a_k \to 0$ as $k \to \infty$.

Then $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

In particular the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

Proof.

The even partial sums $s_{2n} = \sum_{k=1}^{2n}(-1)^{k+1}a_k$ form an increasing sequence, since $s_{2n} - s_{2n-2} = a_{2n-1} - a_{2n} \geq 0$. Similarly the odd partial sums $s_{2n+1} = \sum_{k=1}^{2n+1}$ form a decreasing sequence, since $a_{2n+1} - a_{2n} \leq 0$. Each even partial sum $s_{2n} \leq s_{2n+1} \leq s_1$; each odd partial sum $s_{2n+1} \geq s_{2n+2} \geq s_2$, so both of these sequences of partial sums are bounded and thus converge. Finally, their limits are the same, since $\lim_{n\to\infty}(s_{2n+1} - s_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$. The common limit is the sum of the series. □

Note that this result depends crucially on the order $a_1, a_2, \ldots$ of the terms of the series. It is not surprising that if this order is changed, the series might converge to a different sum (or not converge at all).