Lecture 4-11: Monotone sequences and decreasing series

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On Monday I ended the lecture with the fundamental Monotone Convergence Theorem, which asserts that any bounded increasing sequence converges. Actually a more general result holds. Call a sequence s_n monotone if it is either increasing ($s_n \leq s_{n+1}$ for all n) or decreasing ($s_n \geq s_{n+1}$ for all n)

Theorem 2.4.2, p. 56, revisited

A monotone sequence (s_n) converges if and only if it is bounded.

I first show that every convergent sequence (s_n), monotone or not, is bounded (Theorem 2.3.2, p. 49). Indeed, if s_n converges to L, then there is an index N with $||s_n - L| < 1$ for $n \geq N$, whence also $s_n) < |L| + 1$ for all such n. Letting M be the largest of $|s_1|, \ldots, |s_{n-1}|$, we then get $|s_n| < \max(M, |L| + 1)$ for all n, as desired. It only remains to show that if (s_n) is decreasing and bounded below, then (s_n) converges; this follows from the same argument as in the increasing case, replacing the least upper bound of the set S of terms of (s_n) by the greatest lower bound. Note that the same argument works if instead we have only $s_n \leq s_{n+1}$ for all indices n larger than some N, or $s_n \geq s_{n+1}$ for all n larger than N.

Since the sequences of greatest interest are those of partial sums of an infinite series, I now turn attention to such series. I first consider series $\sum_{i=1}^{\infty} a_i$ where $a_i \geq 0$ for all i; the results extend to the case where $a_i \geq 0$ for all i larger than some N.

Proposition

A series $\sum_{n=1}^{\infty} a_n$ as above converges if and only if its partial sums are bounded.

Proof.

The partial sums s_n form an increasing sequence, which therefore converges if and only if it is bounded. Note that if the s_n are not bounded, then we must have $s_n \to \infty$ as $n \to \infty$. □

There is a very useful criterion for deciding when a series $\sum a_i$ with nonnegative terms has bounded partial sums.

Cauchy condensation test (Theorem 2.4.6, p. 59)

A series $\sum_{n=1}^{\infty} b_n$ with $b_n \geq 0$ for all n converges if and only if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges.

Proof.

If the second series converges, then a partial sum $s_{2^{k+1}-1}$ of the first series equals $b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots \leq b_1 + 2b_2 + \ldots 2^k b_{2^k}$; this last term is the kth partial sum t_{2^k} of the second series. Since the full set of partial sums t_n is bounded, the t_k are also bounded. Since any partial sum s_m of the first series is at most s_{2^k-1} whenever $m < 2^k - 1$, the first series has bounded partial sums and converges. If the second series diverges, then we similarly see that $(b_1+b_2)+(b_3+b_4)+(b_5+b_6+b_7+b_8)+\ldots \geq 2b_2+2b_4+\ldots+2^{k-1}b_{2^k}$. The partial sums of the b_m are not bounded, whence as above the partial sums b_{2^k} are not bounded, and the partial sums of the first series are not bounded, so that the first series diverges. □

Example

The p-series $H_p = \sum_{i=1}^{\infty}(1/i^p)$ converges if and only if $p > 1$. Indeed, the condensed series $\sum_{n=0}^{\infty} 2^n/2^{np} = \sum_{n=0}^{\infty}(2^{1-p})^n$ is geometric and so converges if and only if $2^{1-p} < 1$ or $p > 1$. In particular, the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. By the way, the value of H_p is known for all positive even integers p, but *not* for any odd $p > 1$!

Series $\sum_{n=1}^{\infty}$ with $b_n \geq b_{n+1}$ for all n are often analyzed using something called the integral test, which some of you may have seen. The Cauchy condensation test provides an alternative way to look at such series (since we will not get to integration until much later in the course).

For general (non-monotone) sequences boundedness does *not* imply convergence; e.g. the sequence $s_n = (-1)^n$ is bounded but fails to converge. There is however a very useful way to extract a convergent sequence out of any bounded sequence, whether or not that sequence itself converges. To do this I need a definition.

Definition 2.5.1, p. 62

Given a sequence a_n I say that another sequence b_k is a *subsequence* of a_n if there is a strictly increasing sequence $n_1, n_2, \ldots$ of positive integers such that $b_k = a_{n_k}$. The parameter k indexes the subsequence.

It is easy to see that subsequences of convergent sequences converge to the same limit:

Theorem 2.5.2, p. 63

If (a_n) converges to L, then any subsequence $(b_k = a_{n_k})$ also converges to L.

Indeed, if $a_n \to L$ then for any $\epsilon > 0$ there is N with $|a_n - L| < \epsilon$ whenever $n > N$. Since the indices n_k form a strictly increasing sequence of positive integers, we have $n_k \geq k$ for any k, whence $|b_k - L| < \epsilon$ for any $k \geq N$, as desired. Next I show that any sequence, convergent or not, always has a monotone subsequence.

Theorem

Every sequence a_n has a monotone subsequence.

Proof.

Call an index n a *peak index* if $a_m \leq a_n$ for all $m \geq n$. Then I consider two cases. If there are infinitely many peak indices, say $n_1, n_2, \ldots$ with $n_1 < n_2 < \ldots$, then the definition of peak shows that the subsequence $b_k = a_{n_k}$ is decreasing. If instead there are only finitely many peak indices, then there must be a largest one N (or set $N = 0$ if there are no peak indices at all). Set $n_1 = N + 1$; if $n_1, \ldots, n_k$ have been defined so that $a_{n_1} < \cdots < a_{n_k}$, then since $n_k > N$ there must be an index $m > n_k$ with $a_m > a_{n_k}$; set $n_{k+1} = m$. In this inductive way we have defined indices n_k for all k such that $b_k = a_{n_k}$ is an increasing subsequence of a_n. □

As an immediate corollary every bounded sequence has a convergent subsequence (the Bolzano-Weierstrass Theorem, 2.5.5 on p. 64) since a monotone subsequence of a bounded sequence is again bounded and so must converge.

I conclude by using sequences to define a certain kind of subset of $\mathbb{R}$.

Definition 3.2.7, p. 90

A subset S of $\mathbb{R}$ is *closed* if the limit L of any convergent sequence s_n of points in S also lies in S.

If a set S contains the supremum and infimum of any bounded subset of itself, then it contains the limit of any convergent monotone sequence of its points, whence it contains the limit of any convergent sequence of its points, since any such sequence admits a monotone subsequence with the same limit. Conversely, it is easy to show that a closed set does contain the supremum and infimum of any bounded subset. Thus a closet subset of $\mathbb{R}$ is exactly one for which the Least Upper and Greatest Lower Bound Properties hold.