Lecture 3-31: The real numbers

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Welcome to the wonderful world of real analysis! In this course you will continue your study of calculus, this time acquiring a basic knowledge of the foundations of this subject. Before I turn to its basic material of functions, I need to say a number of things about their domains, namely the set of real numbers; although the basic operations of calculus seem to refer only to functions, they depend crucially on properties of the real numbers. For example, even the identity function $f(x) = x$, if it is regarded as defined only on the set of rational numbers between e and π, has no maximum or minimum on this interval, nor does it even make sense to speak of the derivative or integral of this function on this domain.
All page references will be to the Abbott text "Understanding Analysis", which I have sent to all of you as a pdf. Homework exercises will be largely taken from that book.

I will begin with some basic arithmetic properties of the integers, rational numbers, and real numbers; these pertain to the various ways in which one can combine two of these numbers to produce a third. I denote the set of integers by $\mathbb{Z}$ (standing for "Zahlen", the German word for number). I further denote the set of rational and real numbers by $\mathbb{Q}$, respectively; recall that rational numbers are quotients a/b of integers a, b with $b \neq 0$. Then $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ are all what we call *rings* under addition, subtraction, and multiplication, while $\mathbb{Q}$ and $\mathbb{R}$ have the additional property of being *fields*.

This means that for any a, b lying in any of these sets there are numbers $a+b, a-b, ab$lying in the same set such that $(a+b)+c = a+(b+c), a+b = b+a, a+0 = a$, and $a+(-a) = 0$ for all a, b, c, where $-a$ denotes the negative of a, while similarly $ab = ba, (ab)c = a(bc), a(b+c) = ab+ac$ (see Definition 8.6.4, p. 299). For $\mathbb{Q}$ and $\mathbb{R}$ we further have that if an element a is not zero there is a number $1/a$ in the same set with $a(1/a) = 1$. I denote by a/b the product $a(1/b)$ if $b \neq 0$. Also I denote by $\mathbb{N}$ the set of nonnegative integers; this is closed under addition and multiplication but not subtraction or division.

The fields $\mathbb{Q}$ and $\mathbb{R}$ have an additional important structure; they are *ordered*. Equivalently, one has the sets $\mathbb{Q}^+, \mathbb{R}^+$ of (strictly) *positive* rational and real numbers, respectively, having the the key properties that $a + b, ab \in \mathbb{Q}^+$ whenever $a, b \in \mathbb{Q}^+$, and similarly for $\mathbb{R}^+$ (Definition 8.6.5, p. 299). Moreover, given any $a \in \mathbb{Q}$ with $a \neq 0$, exactly one of $a, -a$ lies in $\mathbb{Q}^+$, and again similarly for $\mathbb{R}^+$. Notice that they do not actually refer to order directly; instead one defines $a < b$ if $a, b \in \mathbb{Q}$ are such that $b - a \in \mathbb{Q}^+$ and similarly for $a, b \in \mathbb{R}$. One writes $a \leq b$ if either $a < b$ or $a = b$. Thus the crucial property defining a total order is satisfied: given a, b with $a \neq b$, either $a < b$ or $b < a$ but not both. If $a < b$ and $c > 0$, then $ac < bc$, while if instead $a < b$ but $c < 0$, then $ac > bc$.

It is in the properties of the order relation that a crucial difference emerges between the rational and real numbers; one can do calculus on the letter but not on the former. To express this difference I need some additional terminology. Call a set S of real numbers bounded above if there is a real number y such that $x \leq y$ for all $x \in S$. I call any such y an upper bound for S. Thus the closed interval $[0, 1]$, consisting of all real numbers x with $0 \leq x \leq 1$, is bounded above, with 1 as an upper bound; the set $\mathbb{N}$ is not bounded above. The key property here is

The Completeness Axiom (p. 15), or the Least Upper Bound Property

Any nonempty set S of real numbers that is bounded above has a least upper bound z; that is, there is an upper bound z for S such that any upper bound w of S has $z \leq w$.

One calls the least upper bound of S its supremum and denotes it by $\sup S$. Note that $\sup S$ may or may not lie in S; for example, the open interval $(0, 1)$, consisting of all x with $0 < x < 1$, and the closed interval $[0, 1]$ both have supremum 1. Note also that both $\mathbb{Z}$ and $\mathbb{N}$ also have the least upper bound property, for trivial reasons; but you will see later that $\mathbb{Q}$ does not; specifically, the set of $x \in \mathbb{Q}$ with $x^2 < 2$ fails to have a least upper bound in $\mathbb{Q}$.

Roughly speaking, this failure means that there are a lot of holes in $\mathbb{Q}$; because of these holes, the limit operation (on which both differentiation and integration crucially depend) does not work well for $\mathbb{Q}$. For example, you are used to the idea that you can write down any decimal expansion, say $1.4142135\ldots$, and have it denote a unique number; but there is no guarantee that that number is rational. If you had never heard of irrational numbers you would therefore be forced to say that decimal expansions like the above one are meaningless; this is clearly not desirable.

I will conclude the lecture today by showing that there is no rational square root of 2. This is usually done by using the properties of even and odd integers under multiplication (see pp. 1 and 2 of the text); but since I have been talking about the order of the real numbers, I will give a different argument based on that. Denoting a positive square root of 2 as usual by $\sqrt{2}$, suppose for a contradiction that $\sqrt{2}$ is rational. Then there is a positive integer M such that $M\sqrt{2}$ is an integer, whence by a fundamental property of $\mathbb{N}$ there is a least such integer N. Since clearly $1 < \sqrt{2} < 2$ one gets that $N(\sqrt{2}-1) = N\sqrt{2} - N$ is a positive integer less than N, and $N(\sqrt{2}-1)\sqrt{2} = 2N - N\sqrt{2}$ is an integer, contradicting the way N was chosen. Thus one of the most glaring holes in $\mathbb{Q}$ is the absence of a square root of 2. You will show this square root exists in $\mathbb{R}$ in homework this week.