HW #1, due 4-4

Math 327C

1. Let S be a nonempty set of real numbers bounded below, so that there is $x \in \mathbb{R}$ with $x \leq y$ for all $y \in S$. Show that S has an infimum, or greatest lower bound z (denoted $\inf S$), so that $z \leq y$ for all $y \in S$ but no real number larger than z has this property. (Use negatives of numbers to deduce this fact from the Least Upper Bound Property).

2. Let e be a positive real number. Show that $x^2 + \frac{e^2}{x^2} \geq 2e$ for all $x > 0$, with equality if and only if $x^2 = e$, by writing the difference $x^2 + \frac{e^2}{x^2} - 2e$ as the square of a real number.

3. Let $d > 0$ and suppose that $x > 0, x^2 > d$. Setting $y = \frac{1}{2}(x + \frac{d}{x})$, use the previous problem to show that $y < x$ and $y^2 > d$.

4. Given $d > 0$ let $S = \{x \in \mathbb{R} : x > 0, x^2 > d\}$. Show that S is nonempty and so has an infimum y. Show that we cannot have $y^2 > d$.

5. For any $z \geq 0$, we have $z^2 > d$ if and only if $(\frac{d}{z})^2 < d$. Using this fact and previous problems, show that the infimum y of the last problem cannot satisfy $y^2 < d$. Deduce that $y^2 = d$, so that every positive real number has a real square root.

Read sections 1.1-4 and 8.6.