MIDTERM #1 SOLUTIONS

1. Show carefully, using the definition of limit, that $\frac{3}{5n^2+3} \to 0$ as $n \to \infty$.

Given $\epsilon > 0$ choose $N \in \mathbb{N}, N > \frac{1}{\sqrt{\epsilon}}$; then for $n \geq N$ we have $|\frac{3}{5n^2+3}| < \frac{3}{5n^2} < \frac{1}{n^2} < \epsilon$, whence $\frac{3}{5n^2+3} \to 0$ as $n \to \infty$.

2. The following are *incorrect* versions of theorems proved in class. In each case give the *correct* statement of the theorem.

(a) A bounded sequence converges if and only if it is monotone.

A monotone sequence converges if and only if it is bounded.

(b) Every nonempty set of real numbers that is bounded below has a least lower bound.

Every nonempty set of real numbers bounded below has a greatest upper bound (of if bounded above has a least upper bound).

(c) If $f(x)^2$ is continuous at $x = a$, then so is $f(x)$.

If $f(x)$ is continuous at $x = a$ then so is $f(x)^2$.

3. Let S be a nonempty set of real numbers that is bounded below and $\{C_x : x \in S\}$ be the corresponding set of cuts of rational numbers. Work out an expression for C, the cut corresponding to the greatest lower bound of S, in terms of the C_x.

The cut C is the intersection of the cuts C_x with the largest element removed if it has one.

4. Show that $\sum_{i=1}^{\infty} \frac{\cos^2 i}{2^i}$ converges, by showing that its partial sums are bounded.

A typical partial sum $\sum_{i=1}^{n} \frac{\cos^2 i}{2^i}$ is bounded by $\sum_{i=1}^{n} \frac{1}{2^i} < 1$, so the partial sums are bounded and the series converges.

5. Let g be a continuous function on $\mathbb{R}$ sending the interval $I = [0, 1]$ to itself. Use the Intermediate Value Property to show that there is $x \in [0, 1]$ with $g(x) = x$.

The function $h(x) = g(x) - x$ is continuous on $[0, 1]$ and takes a nonnegative value at 0 and a nonpositive value at 1, so it must take the value 0 for some $x \in [0, 1]$ and $g(x) = x$, as desired.