Lecture 6-2: Review

June 2, 2023

In this lecture I briefly review once again the convergence tests for series and conclude with the material on integration.

- Comparison Test: $\sum a_n$ converges if $a_n \leq b_n$ for all but finitely many n and $\sum b_n$ converges; $\sum a_n$ diverges if $a_n \geq b_n$ for all but finitely many n and $\sum b_n$ diverges.
- Limit Comparison Test: If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is finite and nonzero, then $\sum a_n$ converges if and only if $\sum b_n$ does.
- Ratio Test: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equals L, then $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$.
- p-series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Given a series $\sum a_n$ for which we do *not* have $a_n \geq 0$ for all but finitely many n, we have

- *n*th term Test: $\sum a_n$ diverges if $\lim_{n\to\infty} a_n$ either fails to exist or exists but has a nonzero value.
- Absolute Convergence Test: $\sum a_n$ converges whenever $\sum |a_n|$ does.
- Alternating Series Test: $\sum (-1)^n a_n$ converges whenever $a_1 \geq a_2 \geq \ldots$ and $a_n \to 0$ as $n \to \infty$.

The Taylor series at $x = a$ of the function $f(x)$ is the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(a)$ denotes the nth derivative of f at a. Thus f must be infinitely differentiable at a for this series to make sense, and indeed it defines an infinitely differentiable function whenever it converges. Such a series can be integrated or differentiated term by term within its radius of convergence. Often we can change variables to derive new Taylor series from old ones. Thus given the series for $\sin x$, namely $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, we get the series for $\sin x^3$ by simply replacing x throughout by x^3 in the above series. We get $\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!}$. Integrating term by term, we get the series for $\int_0^x \sin t^3\, dt$, even though there is no formula for this last function. This series is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+4}}{(6n+4)(2n+1)!}$ All of these series converge absolutely for all x.

Now I review the definition of the integral of a function. Let f be a function bounded on a closed interval $[a, b]$. For every partition P of $[a, b]$, that is, for every finite set of points $\{x_0, \ldots, x_n\}$ with $x_0 = a < x_1 < \ldots < x_n = b$, define the upper sum ($U(f, P)$ and lower sum $L(f, P)$ by $\sum_{i=0}^{n-1} M_i(x_{i+1} - x_i), \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$, respectively, where M_i is the supremum of f on $[x_{i-1}, x_i]$ and m_i is the infimum of f on the same interval. Then we have $L(f, P) \leq U(f, Q)$ for all partitions P, Q. If there is a unique number I such that $L(f, P) \leq I \leq U(f, P)$ for all partitions P, then we say that f is integrable on $[a, b]$ and write $I = \int_a^b f(x)\, dx$. In general, we write $\underline{\int}_a^b f(x), dx, \overline{\int}_a^b f(x)\, dx$, respectively, for the supremum of all $L(f, P)$ and the infimum of all $U(f, P)$, calling these numbers the lower and upper integrals of f over $[a, b]$. To check your understanding of this definition, you might want to consider the function f defined by $f(x) = \frac{1}{|n|}$ if $x = \frac{m}{n} \in \mathbb{Q}$, with $\frac{m}{n}$ in lowest terms, while $f(x) = 0$ if $x \notin \mathbb{Q}$. Decide whether this function is integrable over $[0, 1]$ and if so what its integral is.

If f is continuous on $[a, b]$, or more generally even if f is just assumed to be integrable on $[a, b]$, then we have $I = \int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(y_i)$, where y_i is any point in the closed interval $[a + i(\frac{b-a}{n}), a + (i+1)(\frac{b-a}{n})]$. Given any $\epsilon > 0$, for all sufficiently large n the given sum (called a Riemann sum for f) is guaranteed to fall within ϵ of $\int_a^b f(x)\,dx$, for any choice of y_i. For particular computations the choice $y_i = a + i(\frac{b-a}{n})$ may be especially convenient. Given a limit of sums that an be interpreted as a Riemann sum of a suitable function, you should be able to recognize this function and use the Fundamental Theorem of Calculus to evaluate its integral, thereby computing the original limit.

The strengthened form of the Fundamental Theorem of Calculus that you learned this term is that if f is integrable on $[a, b]$ and continuous at $x \in [a, b]$, then f is also integrable on $[a, x]$ and if we set $F(y) = \int_a^y f(t)\, dt$ for all $y \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$. Even more generally, if you assume only that f is integrable on $[a, b]$, then the function $F(y)$ is continuous.

Finally, a reminder that the final exam is scheduled for Monday, June 5, at 2:30, in the usual classroom Loew 201. Logistics are the same as for the midterms: you are allowed both sides of one sheet of handwritten notes and all work is done on the test paper. Good luck and have a good summer!