Lecture 5-8: Power and Taylor series

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Last time I started to cover infinite series $\sum_{k=0}^{\infty} f_k(x)$ of functions defined on an interval I, showing that if there are positive constants $M_0, M_1, \ldots$ such that $\sum_{k=0}^{\infty} M_k$ converges and $|f_k(x)| \leq M_k$ on I, then $\sum_{k=0}^{\infty}$ converges uniformly to its sum $f(x)$ on I, so that in particular $f(x)$ is continuous on I if the f_k are.

Now I want to consider the most important special case, namely the one in which there is a constant a with $f_k(x) = a_k(x-a)^k$ for some constant a_k. I call the series $\sum_{k=0}^{\infty} a_k x^k$ a Taylor series in x; in the special case $a = 0$ it is called a power series. Any such series converges trivially at $x = a$ to a_0, thanks to a convention for Taylor series that $0^0 = 1$. The first result is that Taylor series cannot flip wildly back and forth between convergence and divergence as $|x-a|$ increases from 0.

Interval of Convergence Theorem: Proposition 9.40, p. 258

If $\sum_{k=0}^{\infty} a_k(x-a)^k$ converges at $x = a + R$, then it converges absolutely and uniformly for $[a-M, a+M]$ for any M with $0 \leq M < |R|$.

Proof.

If $\sum_{k=0}^{\infty} a_k R^k$ converges, then in particular $a_k R^k \to 0$ as $k \to \infty$, whence one has $|a_k x^k| = |a_k R^k||(\frac{x}{R})| \leq (\frac{M}{|R}|)^k$ for k sufficiently large and $\sum_{k=0}^{\infty} |a_k x^k|$ converges uniformly on $[-M, M]$ by the Weierstrass M-test. □

As a consequence, given any Taylor series $\sum_k a_k(x-a)^k$, either it converges absolutely for all $x \in \mathbb{R}$, or it converges only for $x = a$, or there is some $R > 0$ such that the series converges for any $x \in (a-R, a+R)$ but diverges for any x with $|x-a| > R$. The series may or may not converge, and if it does converge may do so absolutely or conditionally, at $x = a + R$ and $x = a - R$; these cases are deliberately left ambiguous.

In the first case one says that $\sum_k a_k x^k$ has infinite radius of convergence; in the second case it has radius of convergence 0. In the last case one says that this series has radius of convergence R.

Example

The series $\sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}$ has infinite radius of convergence. The series $\sum_{k=0}^{\infty} k!(x-a)^k$ radius of convergence 0. The series $\sum_{k=0}^{\infty}(x-a)^k$ has radius of convergence 1, diverging at both endpoints $x = a - 1$ and $x = a + 1$. The series $\sum_{k=1}^{\infty} \frac{(-1)^k(x-a)x^k}{k}$ also has radius of convergence 1; this last series converges at $x = a + 1$ but diverges at $x = a - 1$.

There is a uniform formula for the radius of convergence of any Taylor series $\sum_{k=0}^{\infty} a_k x^k$, which makes use of the limit superior of a sequence.

Formula for Radius of Convergence: compare Exercise 14, p. 263

The radius of convergence R of a Taylor series $\sum_{k=0}^{\infty} a_k (x-a)^k$ is given by the formula $R = \frac{1}{\overline{\lim}_{k\to\infty} |a_k|^{1/k}}$, interpreting $\frac{1}{0}$ as ∞ and $\frac{1}{\infty}$ (for this purpose only) as 0.

This follows at once from the Root Test for convergence (proved in HW). The ambiguous cases of $x = a \pm R$, which are not settled by this test, are irrelevant to the definition of the radius of convergence. There is a similar test based on the Ratio Test, which is less powerful but often easier to apply: if $\lim \frac{|a_k|}{|a_{k+1}|}$ exists (allowing ∞ as a possible value), then R equals this limit. Note that the ratio here is $|\frac{a_k}{a_{k+1}}|$, *not* $|\frac{a_{k+1}}{a_k}|$ as in the ratio test.

I now digress a bit from Taylor series to discuss term-by-term differentiation of sequences of functions in general. As mentioned previously, there is no guarantee in general that even the uniform limit f of a sequence of differentiable functions is differentiable at any point. We will see later that in fact *any* continuous function on a closed bounded interval is the uniform limit of polynomial functions on this interval and that there are continuous functions that are not differentiable at any point. Nevertheless, we have the following result.

Theorem 9.33, p. 252

Let f_n be a pointwise convergent sequence of functions with continuous derivatives on an interval $I = [a, b]$ such that the sequence f_n' converges uniformly on I, say to g, and let $f(x) = \lim_{n\to\infty} f_n(x)$. Then f_n converges uniformly on I to f, f is differentiable on I, and we have $f' = g$ on I.

Proof.

Integrability of uniformly convergent sequences guarantees that $\lim_{n\to\infty}(f_n(x) - f_n(x_0)) = \lim_{n\to\infty}\int_{x_0}^{x} f_n'(x)\,dx = f(x) - f(x_0) = \int_{x_0}^{x} g(t)\,dt$ for $x_0, x \in I$. Now the result follows at once from the Fundamental Theorem of Calculus, together with the continuity of g. □

Proposition 9.40, p. 258

Given a Taylor series $\sum_{n=0}^{\infty} a_n(x-a)^n$ with a positive (or infinite) radius of convergence R, its sum f is differentiable on $(a-R, a+R)$, with derivative $f'(x)$ given by the term-by-term differentiated series $\sum_{n=1}^{\infty} n a_n(x-a)^{n-1}$. We also have $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$ for $|x-a| < R$.

Proof.

The second assertion follows at once since integration commutes with uniform limits. We have $\overline{\lim}_{n\to\infty}|a_n|^{1/n} = \overline{\lim}_{n\to\infty}((n+1)|a_{n+1}|)^{1/n} = \overline{\lim}_{n\to\infty}(\frac{|a_{n-1}|}{n})^{1/n}$, so that all three series have the same radius of convergence. Then the first assertion follows at once from the previous theorem. □

Starting from the geometric series, which is the only Taylor series we know how to sum at this point, we can use this result to vastly enlarge our repertoire of series with known sums. For example, we have $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x)$ for $|x| < 1$. Note in this case that the series converges as $x = -1$ even though the geometric series does not. We would expect that $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = -\ln(2)$; but note that we cannot yet prove this with the tools we have so far. I will state a result after the midterm that will justify this formula.