Lecture 5-31: Review

May 31, 2023

I will begin with a quick review of a fundamental construction that came up in the first week of class. Recall that every real number x is defined as the corresponding cute C_x, consisting by definition of all *rational* numbers strictly less than x. Thus whenever $y \in \mathbb{Q}$ lies in a cut C_x and $z \in \mathbb{Q}$ with $z < y$, then $z \in C_x$; no cut C_x has a largest number; and no cut C_x is either empty or all of $\mathbb{Q}$.

With this definition we see that the supremum (or least upper bound) of a nonempty set S of real numbers bounded above is just the union of the cuts corresponding to the elements of S; the simple set-theoretic construction of taking the union realizes the supremum of a bounded set. Similarly, the infimum of S (if it is nonempty and bounded below) is just the intersection of the cuts corresponding to the elements of S, with the largest element of this intersection removed if it has one.

Once we know that nonempty sets of numbers that are bounded above have suprema, the main criterion telling us when a sequence must converge even if we don't know its limit follows: a monotonic sequence converges if and only if it is bounded. This result about sequences provides the basis for all the convergence tests for series that were reviewed before the second midterm.

Having briefly treated sequences I quickly moved on to functions (before discussing series). The single most important definition is that of continuity at a point: a real-valued function f defined on an interval including a point a is continuous at a if and only if $f(x) \to f(a)$ as $x \to a$, so that for every $\epsilon > 0$ there is $\delta > 0$ such that if $|x - a| < \delta$ and $f(x)$ is defined, then $|f(x) - f(a)| < \epsilon$. Equivalently, f is continuous at a if and only if for all sequences $a_n \to a$ such that $f(a_n)$ is defined for all n we have $f(a_n) \to f(a)$.

Functions continuous on (i.e. at every point of) closed bounded intervals $[a, b]$ are especially nice: any such function is bounded on the interval, takes on a maximum and minimum value, and takes on every value in between. More succinctly, we must have $f[a, b] = \{f(x) : x \in [a, b]\} = [c, d]$ for some c, d with $c \leq d$.

In particular, the rather exotic equation $x^5 - x = \sin x$ must have a solution between 1 and 2; since the continuous function $g(x) = x^5 - x - \sin x$ is negative at $x = 1$ but positive at $x = 2$, it must be 0 at some point between 1 and 2 (though there is no formula for this point).

There is a result of a similar flavor for for differentiable functions, namely the Mean Value Theorem: given any function f continuous on an interval $[a, b]$ and differentiable on the open interval (a, b) there is $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. In particular, for example, if such a function f takes the same value say five times on $[a, b]$, then f' must take the value 0 at least four times on this interval.

There are two results both called the Inverse Function Theorem, one pertaining to continuous functions, the other to differentiable ones. The first asserts that if f is continuous and one-to-one on an interval $[a, b]$, taking it to the interval $[c, d]$, then the inverse function $g =^{-1}$ on $[c, d]$, taking it to $[a, b]$, is also continuous. The second asserts that if in addition f is differentiable $[a, b]$ and its derivative is never 0 there, then g is also differentiable; we have $g'(f(x)) = 1/f'(x)$ for $x \in [a, b]$.

Next time I will review infinite series; I will be brief since this material was reviewed just before the second midterm. I will then spend the rest of the time on integration.