Lecture 5-3: Infinite series, continued

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I begin by mentioning that there are an enormous number of conditionally convergent series, far more than the Alternating Series Test alone can provide. The next theorem massively generalizes this test.

Abel's Theorem

Let $\sum_{i=1}^{\infty} a_i b_i$ be a series such that $a_i \geq a_{i+1}$ for all i, $a_i \to 0$ as $i \to \infty$ and such that the partial sums of the series $\sum_{i=1}^{\infty} b_i$ are bounded (one does *not* need this series to converge). Then $\sum_{i=1}^{\infty} a_i b_i$ converges.

The Alternating Series Test is the special case where $b_i = (-1)^i$; but note for example that one could replace the alternating pattern of signs here by any even-periodic pattern, so that $b_i = \pm 1$ for all i and there is k with $b_i = b_{i+2k}$ for all n, provided that there are the same number of 1s and -1s among $b_1, \ldots, b_{2k}$.

This last result can be used to show that the series $\sum_{n=1}^{\infty} a_n \sin nx$ converges for all $x \neq 0$, provided that the sequence a_n of coefficients decreases to 0 as $n \to \infty$, thanks to a clever formula for the partial sums of $\sum_{n=0}^{m} \sin nx$ which shows that this sum is bounded in m for any $x \neq 0$. Series of this sort are called Fourier series and are of considerable importance in physics and engineering as well as mathematics. They can for example be used to compute the sum of the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ for any positive even integer p.

Now I explore conditionally convergent series in more detail. Given one such, say $\sum_{n=1}^{\infty} a_n$, set $p_n = a_n$ if $a_n \geq 0$, while $p_n = 0$ otherwise; similarly set $q_n = -a_n$ if $a_n \leq 0$ and $q_n = 0$ otherwise. It is easy to check that $p_n, q_n \geq 0$ for all n and both series $\sum_n p_n$ and $\sum_n q_n$ diverge in this situation, since $\sum_n a_n$ does not converge absolutely. On the other hand, convergence of $\sum_n a_n$ forces both $p_n \to 0$ and $q_n \to 0$ as $n \to \infty$.

A particular example is the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$; here $p_n = \frac{1}{2n-1}$, $q_n = \frac{1}{2n}$. Recall that I showed in an earlier lecture that it is possible to rearrange the terms of this series so that its sum gets multiplied by $3/2$. Now I will prove a remarkable generalization of this example.

Rearrangement Theorem

The terms of a conditionally convergent series $\sum a_n$ may be rearranged to make the series converge to any desired sum S.

Proof.

Given S, let N_1 be the smallest index such that $\sum_{n=1}^{N_1} p_n > S$; such an N_1 exists since $\sum p_n$ diverges. The choice of N_1 guarantees that $s_1 = \sum_{n=1}^{N_1} p_n$ lies between S and $S + p_{N_1}$. Take $p_1, \ldots, p_{N_1}$ to be the first N_1 terms of the rearranged series. Now choose the least index N_2 with $s_2 = s_1 - \sum_{n=1}^{N_2} q_n < S$; such an N_2 exists since $\sum_n q_n$ diverges. Then we have $s_2 > S - q_{N_2}$. Take $-q_1, \ldots, -q_{N_2}$ to be the next N_2 terms of the rearranged series. Now $\sum_{n=N_1+1}^{\infty} p_n$ still diverges, so there is a least index $N_3 > N_1$ with $s_3 = s_2 + \sum_{n=N_1+1}^{N_3} p_n > S$, so that $s_3 < S + p_{N_3}$. Take $p_{N_1+1}, \ldots, p_{N_3}$ to be the next block of terms in the rearranged series. $\square$

Proof.

Continue in this way, adding previously unused negative terms until the partial sum first dips below S, then adding previously unused positive terms until this sum first goes above S, and so on. The upshot is that all partial sums of the rearranged series lie within a suitable single term p_n or q_n of S. Now given any $\epsilon > 0$, there is an index N such that $p_n, q_n < \epsilon$ for $n \geq N$; the construction shows that all partial sums of the rearranged series lie within ϵ of S provided they include all terms $p_1, \ldots, p_N, q_1 \ldots, q_N$, as they do past a certain point. Thus the rearranged series converges to S, as desired. □

This result is quite startling when you first see it, but as you gain experience it will become less surprising; the idea is simply that since the positive terms can push the sum as far to the right as desired, and similarly the negative terms can push the sum as far to the left as desired, one can strike an arbitrary balance between these terms.

On the other hand, rearrangements of terms have no effect on the sums of absolutely convergent series.

Theorem

Any rearrangement of the terms of the absolutely convergent series $\sum a_n$ converges to the same sum.

Proof.

If $\sum_{n=1}^{\infty} a_n$ converges absolutely then the subseries $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ defined above also converge. Any rearrangement of $\sum_{n=1}^{\infty} p_n$ converges to the same sum, since if $\sum p_n = S, \epsilon > 0$, and N is chosen large enough that $\sum_{n=1}^{N} p_n > S - \epsilon$, then any rearrangement $\sum p'_n$ of $\sum p_n$ has all of $p_1, \ldots, p_N$ occurring among $p'_1, \ldots, p'_M$ for sufficiently large M, whence the partial sum $\sum_{n=1}^{r} p'_n$ also lies within ϵ of S for sufficiently large r. Similarly any rearrangement of $\sum q_n$ converges to the same sum, and the result follows. □

Following Chapter 9, I now turn to sequences and series of functions $f_n(x)$ rather than numbers a_n. These provide a powerful tool not only for studying the convergence of many sequences or series of numbers at the same time (one for every value of the variable x) but also (given additional tools from calculus) working out the sums of a number of series. I already gave the example $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges absolutely for any $x \in \mathbb{R}$, by the ratio test. Similarly the ratio test also shows that the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converge absolutely for any $x \in \mathbb{R}$. I will identify the sums of these series later.