Lecture 5-26: Riemann integration, concluded

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I will wrap up my treatment of Riemann integration with a few words about improper integrals (that is, integrals such that either the integrand is unbounded or the interval of integration is infinitely long, or both).

Definition

Given a point a and a function f integrable on $[a, b]$ for all $b > a$, we write $\int_a^\infty f(x)\,dx = L$ if the limit of $\int_a^b f(x)\,dx$ as $b \to \infty$ exists and equals L. If instead f is integrable on $[c, b]$ for all $c \in (a, b)$ (with b fixed) then we write $\int_a^b f(x)\,dx = L$ if the limit of $\int_c^b f(x)\,dx$ exists as $c \to a^+$ and equals L; we make a similar definition if $\int_a^c f(x)\,dx$ exists for all $c \in (a, b)$, taking the limit of $\int_a^c f(x)\,dx$ as $c \to b^+$.

Example

The function $f(x) = x^r$ has antiderivative $\frac{x^{r+1}}{r+1}$ if $r \neq -1$; it follows at once that $\int_1^\infty x^r\,dx$ exists if and only if $r < -1$ and equals $\frac{-1}{r+1}$ in this case. Similarly, $\int_0^1 x^r\,dx$ exists if and only if $r > -1$ and equals $\frac{1}{r+1}$ in this case. The integral $\int_0^\infty x^r\,dx$ never exists; the integrand always "runs into trouble", either at 0 or ∞. Also $\int_0^\infty \frac{dx}{1+x^2}\,dx = \frac{\pi}{2}$.

As with series, there is a notion of absolute convergence for integrals: we say that $\int_a^\infty f(x)\,dx$ converges absolutely if $\int_a^\infty |f(x)|\,dx$ converges and this implies that $\int_a^\infty f(x)\,dx$ converges. Also the Comparison and Limit Comparison Tests apply. If we have $f(x) \geq g(x) \geq 0$ on $[a,\infty)$ and if $\int_a^\infty f(x)\,dx$ converges, then so does $\int_a^\infty g(x)\,dx$. If instead $f(x), g(x) \geq 0$ and if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is nonzero and finite, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^\infty g(x)\,dx$ does. The same tests apply for the other kind of improper integral, where $f(x), g(x)$ blow up at $x = a$ (or $x = b$), but at no other point in (a,b). For example, $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ converges: the integrand $\frac{1}{\sqrt{1-x^2}}$ is bounded above by $\frac{1}{\sqrt{1-x}}$, and $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = (-2)(1-x)^{1/2}|_0^1 = 2$.

This last integral is especially interesting because once we know that it converges we can define the number π via $\pi/2 = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$, without any use of geometry or trigonometry.

Again as with series, there is a notion of conditional convergence for integrals: if $\int_a^\infty f(x)\,dx$ converges but does not converge absolutely, then we say it converges conditionally. A classical example of a conditionally convergent improper integral is $\int_\pi^\infty \frac{\sin x}{x}\,dx$. Here integration by parts shows that $\int_\pi^\infty \frac{\sin x}{x}\,dx = \frac{-\sin x}{x^2}|_\pi^\infty + \int_\pi^\infty \frac{\cos x}{x^2}\,dx$; since this last integral converges absolutely, the first integral converges. It does not converge absolutely, since for any x lying within $\pi/3$ of an odd multiple $k\pi/2$ of $\pi/2$ we have $|\frac{\sin x}{x}| > \frac{1}{(k+1)\pi}$, whence $\int_\pi^\infty |\frac{\sin x}{x}|\,dx \geq \sum_{n=0}^\infty \frac{2\pi}{3\pi(2n+2)}$ and the series diverges. The integral $\int_0^\infty \frac{\sin x}{x}\,dx$ can be shown to converge to $\pi/2$.

In fact we have an analogue of Abel's Theorem for integrals. Recall that Abel's Theorem for series states that $\sum a_n b_n$ converges (usually conditionally) if $a_1 \geq a_2 \geq \ldots$, $a_n \to 0$ as $n \to \infty$, and $\sum b_n$ has bounded partial sums. Using partial integrals $\int_a^x g(t)\,dt$ one can similarly show that $\int_a^\infty f(t)g(t)\,dt$ converges (again usually conditionally) if $f(t)$ is weakly decreasing, $f(t) \to 0$ as $t \to \infty$, and there is a uniform bound for the partial integrals $\int_a^x g(t)\,dt$ for all $x \geq a$. In the case of the integral $\int_0^\infty \frac{\sin x}{x}\,dx$, if we compare it to the absolutely convergent integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$, we find that the second integrand, unlike the first, is always nonnegative (suggesting that the second integral is larger), but its absolute value is less than that of the first (suggesting that the second integral is smaller). Remarkably enough, these two effects exactly cancel out. The value of $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is also $\pi/2$.

I will conclude with the Integral Test for convergence of series with nonnegative terms (which many of you will have seen; this is proved in the text as Corollary 9.11 on p. 233). Let $f(x)$ be a weakly decreasing function of $x \in \mathbb{R}^+$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ does. In particular, since $\int_1^{\infty} x^p\,dx$ converges if and only if $p < -1$, I deduce that $\sum_{n=1}^{\infty} n^p$ converges if and only if $p < -1$; this recovers the p-series test that I proved earlier using the Cauchy Condensation Test. The Integral Test follows since a typical partial sum $\sum_{n=1}^{m} f(n)$ is bounded between $\int_1^{m} f(x)\,dx$ and $\int_1^{m+1} f(x)\,dx$, so that the partial sums of the series have a finite limit if and only if the partial integrals $\int_1^{x} f(t)\,dt$ do as $x \to \infty$.

A benefit of the Integral Test is that it gives a good indication of just how fast divergent series $\sum_{n=1}^{\infty} f(n)$ diverge. For example, we already know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; but the proof of the Integral Test shows that the difference $\sum_{n=1}^{m} \frac{1}{n} - \int_1^m \frac{1}{x}\,dx = \sum_{n=1}^{m} \frac{1}{n} - \ln m$ decreases with m and has a finite limit as $m \to \infty$. This limit is called the Euler-Mascheroni constant and is usually denoted γ. It is virtually certain that γ is irrational (it is known that if γ is rational and equal to p/q in lowest terms, then q has at least 20800 decimal digits), but no one knows this for sure.