Lecture 5-22: Riemann integration

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For the remainder of the course I will concentrate on (Riemann) integration, as described in Chapter 6 of the text. The basic idea is first to write down a limit defining the area between the graph of a bounded function f on a closed bounded interval $[a, b]$ and the interval $[a, b]$ on the x-axis itself, counting this area as positive whenever the graph of f lies above the x-axis and as negative whenever this graph lies below the axis, then (eventually) to evaluate that limit by antidifferentiating the function f, thereby deducing in particular that every continuous function on $[a, b]$ has an antiderivative (so is the derivative of another function).

To make things as general as possible, we begin with a partition of $[a, b]$, that is, a finite set $P = \{x_0, \dots, x_n\}$ of points $x_0 = a < x_1 < \dots < x_n = b$. Then P divides $[a, b]$ into subintervals $[x_0 = a, x_1], \dots, [x_{n-1}, x_n = b]$. We do not assume that the subintervals $[x_i, x_{i+1}]$ are of equal length. The gap of P is defined to be the largest difference $x_i - x_{i-1}$. For each index i with $1 \le i \le n$ we let m_i, M_i respectively denote the infimum and supremum of f on the interval $[x_{i-1}, x_i]$. It is then intuitively clear that the sums $\sum_{i=1}^n m_i(x_i - x_{i-1}), \sum_{i=1}^n M_i(x_i - x_{i-1})$ are respectively less and greater than the area we are trying to define and evaluate. We denote these sums by $L(f, P), U(f, P)$, respectively, and call them the lower and upper (Darboux) sums of f with respect to P.

I will show shortly that $L(f,P) \le U(f,Q)$ for any partitions P, Q of $[a,b]$; this is Lemma 6.3 on p. 139. It then follows that the supremum of the set of all lower sums $L(f,P)$ and the infimum of all upper sums $U(f,P)$ (as P runs through all partitions of $[a,b]$) both exist. We denote these by $\underline{\int}_a^b f(x)\,dx, \overline{\int}_a^b f(x)\,dx$, respectively, and call them the lower and upper integrals of f (see the definition on p. 140 of the text). We say that f is integrable if its lower and upper integrals coincide; in this case we denote their common value by $\int_a^b f(x)\,dx$. If $f \le g$ on $[a,b]$ then it is clear from the definition that $\underline{\int}_a^b f(x)\,dx \le \underline{\int}_a^b g(x)\,dx$ and similarly with $\underline{\int}$ replaced by $\overline{\int}$; in particular, if both f and g are integrable, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Also f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition P with $U(f, P) - L(f, P) < \epsilon$, since if f is integrable and ϵ is given, we can find partitions P, Q with $U(f, P) - L(f, Q) < \epsilon$; setting $R = P \cup Q$, we get $U(f, R) - L(f, R) \leq U(f, P) - L(f, Q) < \epsilon$, as claimed. Equivalently, f is integrable if and only if there is a sequence P_n of partitions of $[a, b]$ with $\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$. This is the Archimedes-Riemann Theorem (Theorem 6.8, p. 143).

To show that $L(f,P) \leq U(f,Q)$ for any partitions P,Q of $[a,b]$, note first that it is clear from the definition that $L(f,P) \leq U(f,P)$ for all P. Given $P = \{x_0,\ldots,x_n\}$, let P' be obtained from P by adding one new point y, say between x_{i-1} and x_i. Comparing $L(f,P)$ to $L(f,P')$ we find that the single term $m_i(x_i - x_{i-1})$ is replaced by the sum $m_i'(y - x_{i-1}) + m_i''(x_i - y)$, where m_i', m_i'' are the respective infima of f on $[x_{i-1},y]$ and $[y,x_i]$; since $m_i \leq m_i', m_i''$ we get $L(f,P') \geq L(f,P)$; similarly $U(f,P') \leq U(f,P)$. By induction we deduce that $L(f,P) \leq L(f,R)$ whenever the partition R of $[a,b]$ contains P as a set; similarly $U(f,P) \geq U(f,R)$ in this situation (Lemma 6.2, p. 139). But now since the union $P \cup Q$ of P,Q is a partition of $[a,b]$ whenever P,Q are, we deduce that $L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q)$, as desired.

Now consider two extreme examples.

Example

If $f = c$ is a constant function, then it is easy to see that $L(f, P) = c(b - a) = U(f, P)$ for all partitions P of $[a, b]$, whence f is integrable on $[a, b]$ and $\int_a^b f(x)\,dx = c(b - a)$. On the other hand, if $f(x) = 0$ for $x \in \mathbb{Q}$, $f(x) = 1$ for $x \notin \mathbb{Q}$, then for any interval $[a, b]$ we have $L(f, P) = 0, U(f, P) = b - a$ for all partitions P, since every subinterval $[x_{i-1}, x_i]$ contains at least one rational and at least one irrational number. Thus $\underline{\int}_a^b f(x)\,dx = 0, \overline{\int}_a^b f(x)\,dx = b - a$, so that f is not integrable on $[a, b]$. See Examples 6.5 and 6.6 on pp. 140 and 141.

Many functions are integrable. In particular we have

Theorem 6.18, p. 156

Any continuous function f on an interval $[a, b]$ is integrable.

Proof.

Given $\epsilon > 0$, use the uniform continuity of f (Theorem 3.17, p. 68) to choose $\delta > 0$ so that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ whenever $x, y \in [a, b], |x - y| < \delta$. Then given any partition $P = \{x_0, \ldots, x_n\}$ whose gap is less than δ (for example the regular partition $P_n = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, b\}$ for sufficiently large n) we have $U(f, P) - L(f, P) \leq \frac{\epsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon$, whence f is integrable by the Archimedes-Riemann Theorem. $\square$

In fact, if f is continuous and for each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ we choose a point $y_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$, then the Riemann sum $R(f, P) = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1})$ converges to $\int_a^b f(x)\,dx$ as the gap of P goes to 0, in the sense that for every $\epsilon > 0$ there is $\delta > 0$ such that $|R(f, P) - \int_a^b f(x)\,dx| < \epsilon$ whenever the gap of P is less than δ; it does not matter at all how the points y_i are chosen. In fact, by working a little harder, we can show that this is true for any integrable function f on $[a, b]$, even if it is not continuous (see Chapter 7 in the text). Note however that integrability is crucial here: the non-integrable function f in the last example is 0 at every rational number, so any Riemann sum $R(P_n, f)$ with a regular partition P_n of the unit interval $[0, 1]$ with the midpoint of each interval $[x_{i-1}, x_i]$ chosen as the point y_i equals 0; but the integral $\int_0^1 f(x)\,dx$ is not defined.