Lecture 5-17: Review

May 17, 2023

I will begin by reviewing the definition of continuity: a function f defined on an interval (a, b) containing a point c is continuous at a point c if $\lim_{x\to c} f(x) = f(c)$, or equivalently given any sequence (c_n) of points converging to c such that $f(c_n)$ is defined for all n, we have that $(f(c_n))$ converges to $f(c)$. If instead f is defined e.g. on an interval $[a, b]$ and $c = a$, then we say that f is continuous at a if $\lim_{x\to a^+} = f(a)$; a similar definition holds for $c = b$. If $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists for all $c \in [a, b]$ (replacing this limit by the appropriate one-sided limit if $c = a$ or $c = b$), then we say that f is differentiable on $[a, b]$ and write $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ for its derivative at c.

The three most important properties of a continuous function f on a closed bounded interval $[a, b]$ can be neatly captured in the single statement that $f[a, b]$, the image $\{f(x) : x \in [a, b]\}$ of the interval $[a, b]$ under f, is another closed bounded interval $[m, M]$ for some m and M.

Thus in particular f is bounded on $[a, b]$ (Boundedness Theorem); it takes on a maximum and a minimum on $[a, b]$ (Extreme Value Theorem); and it takes on every value between its maximum and minimum (Intermediate Value Theorem). Going beyond the text, I also showed that any derivative f' of a differentiable function on an interval $[a, b]$ also satisfies the Intermediate Value Theorem, whether or not it is continuous on this interval; but it need *not* satisfy either the Boundedness Theorem or the Extreme Value Theorem.

The most important theoretical result about derivatives is the Mean Value Theorem, which asserts that if f is continuous on $[a, b]$ and differentiable on (a, b) then there is some $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. This is used to show, for example, that such an f is weakly increasing on $[a, b]$ if and only if $f'(x) \geq 0$ for all $x \in [a, b]$.

Turning now to infinite series, recall first that a series $\sum a_n$ with nonnegative terms a_n converges if and only if its partial sums are bounded. This leads to the following battery of convergence tests. Throughout we assume that $a_n, b_n \geq 0$ for all but finitely may indices n.

- Comparison Test: $\sum a_n$ converges if $a_n \leq b_n$ for all but finitely many n and $\sum b_n$ converges; $\sum a_n$ diverges if $a_n \geq b_n$ for all but finitely many n and $\sum b_n$ diverges.
- Limit Comparison Test: If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is finite and nonzero, then $\sum a_n$ converges if and only if $\sum b_n$ does.
- Ratio Test: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equals L, then $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$.
- p-series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Given a series $\sum a_n$ for which we do *not* have $a_n \geq 0$ for all but finitely many n, we have

- *n*th term Test: $\sum a_n$ diverges if $\lim_{n\to\infty} a_n$ either fails to exist or exists but has a nonzero value.
- Absolute Convergence Test: $\sum a_n$ converges whenever $\sum |a_n|$ does.
- Alternating Series Test: $\sum(-1)^n a_n$ converges whenever $a_1 \geq a_2 \geq \dots$ and $a_n \to 0$ as $n \to \infty$.

A Taylor series (at $x = a$, where a is a constant) is an infinite series of the form $\sum a_n(x-a)^n$. A Taylor series is called a power series if $a = 0$. Any Taylor series has a radius of convergence R, which is either $0, \infty$, or a positive real number. If $R = 0$, then the series converges only at $x = a$, while if $R = \infty$, then the series converges for all x. If R is finite and nonzero, then the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$; either behavior is possible for $|x-a| = R$. We have $R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$ whenever this limit exists, where for this purpose we allow ∞ as a limit.

In addition to abstract Taylor series, we have *the* Taylor series at $x = a$ of an (infinitely differentiable) function f, which is defined to be $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. This may or may not converge to $f(x)$ in general. The three most important special cases where it does converge to $f(x)$ for all x, all with $a = 0$, are the series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$; another important series is $-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$. This last series is valid for $x \in [-1, 1)$.

Partial sums of Taylor series are a special case of convergent sequences (f_n) of functions f_n on an interval I, which are sequences f_n such that the sequence $f_n(x)$ of numbers converges for all $x \in I$. Writing $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in I$, we call f the pointwise limit of the f_n. If (f_n) converges uniformly on I, so that (by definition) for every $\epsilon > 0$ there is an index N with $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in I$, then f is better behaved than in general: in this situation f is continuous if the f_n are and furthermore $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$. A series $\sum_{n=1}^{\infty} f_n$ of functions f_n converges uniformly on an interval I if there are constants M_1 for $i \geq 1$ such that $|f_i(x)| \leq M_i$ for all i and $\sum M_i$ converges (the Weierstrass M-test). Unfortunately, even the uniform limit of differentiable functions need not be differentiable. Likewise, the pointwise limit of continuous functions need not be continuous.

Logistics are the same as for the last exam; you are allowed one sheet of notebook paper with handwritten notes on front and back and all work will be done on the test paper. The midterm will cover Chapter 3 and 4 and the first four sections of Chapter 9 in the text, together with the fact about derivatives on closed bounded intervals mentioned above.