Lecture 5-15: Fourier series

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I will conclude my treatment of series with a brief account of Fourier series, that is, of functions defined by series of the form $\sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$. You will see that these series behave very dfiferently than power series. Just as one cannot truly understand the grammar of one's native language until one studies a foreign one, so one cannot truly understand power series until one comes to grips with how infinite series of functions behave in a very different setting.

My goal is to express a continuous function $f(x)$ as the sum of a convergent series $\sum_{n=0} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx)$. The first thing to be said about this goal is that it cannot be achieved, even for a very simple function like $f(x) = x$! This is because any Fourier series, being a sum of periodic functions with period 2π, can converge only to a periodic function, which the identity function x is not. To fix this problem, begin by restricting to the open interval $(-\pi, \pi)$. Then $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx$ does converge to x on this interval, though not at either endpoint $\pm\pi$ (it converges to 0 at both of those points).

What's going on here? The short answer is that the series $\sum_{m=0}^{\infty} \frac{2}{2m+1} \sin(2m+1)x$ is doing its best to converge to x on $[-\pi, \pi]$, but since it has to take the same value at π as at $-\pi$, it in effect makes a compromise, converging to $0 = (\pi + (-\pi))/2$ there. In general, Fourier series have to be restricted to the interval $[-\pi, \pi]$. Then it turns out that any differentiable function $f(x)$ on $[-\pi, \pi]$ admits a Fourier series expansion on that interval, converging to $f(x)$ if $x \neq \pm\pi$ and to $\frac{f(\pi)+f(-\pi)}{2}$ at $\pm\pi$. But how can an infinite series of continuous functions on $[-\pi, \pi]$ converge to a discontinuous one at $\pm\pi$? The answer is that the convergence is not uniform! Nor is it absolute in general: plugging in $x - \pi/2$ into the series for x gives twice the alternating odd harmonic series, or $2\sum_{m=0}^{\infty} \frac{(}{-} 1)^m 2m + 1$, which as we now know converges to $\pi/2$, but only conditionally.

The coefficient a_n of $\cos nx$ in the Fourier series of a continuous function $f(x)$ is given by $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ if $n = 0$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx$ for $n > 0$; the coefficient b_n of $\sin nx$ in this series equals $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ in all cases; note that there is no point in including the 0 function $\sin 0x$ as a term but the term $\cos 0x = 1$ can occur, so that the a_n start at $n = 0$ but the b_n start at $n = 1$. Note also that these formulas make sense for any continuous function, unlike the formula $f^{(n)}(a)/n!$ for the coefficient of $(x-a)^n$ in the Taylor series of f, which requires that f be infinitely differentiable at a.

Thus many more functions can be represented as Fourier series than Taylor series, though the price one pays for this is the convergence of a typical Fourier series is much slower than that of a Taylor series. Note also that the periodicity of the terms $a_n \cos nx$, $b_n \sin nx$ of the terms in a Fourier series imply that such a series is neither more nor less apt to converge at a large x than a small one; there is no notion of radius of convergence for a Fourier series.

Another stark difference between Fourier and Taylor series emerges emerges from the following example. You have seen that the Fourier series for x involves only $\sin nx$ terms; this is not surprising, since both x and $\sin nx$ are odd functions for any n. LIkewise the series for the even function $|x|$ involves only $\cos nx$ terms: it is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}$. But x and $|x|$ are the same function for $x \geq 0$; we thus get *two* Fourier series for the same function on $[0, \pi]$.

Such behavior cannot occur for Taylor series, since the only Taylor series at $x = a$ that can possibly converge to a function f is the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. Also observe that Fourier series cannot generally be differentiated or integrated term by term: differentiating the series $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx$ term by term gives the series $\sum_{n=1}^{\infty} 2(-1)^{n-1} \cos nx$, which does not converge for any x.

Plugging in $x = \pi$ to the series for $|x|$ and observing that $|-\pi| = |\pi| = \pi$, I deduce that $4\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{2}, \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$; since $\sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$. I get $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, a famous formula due to Euler which actually calculates the sum of the p-series at $p = 2$. More generally, one can use Fourier series to work the sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for any positive even integer p; the answer turns out to be π^p times a rational number. But already for $p = 3$, this sum remains a total mystery after three centuries; the best one can do with Fourier series is to show that the alternating sum $\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3}$ of the odd reciprocal cubes is $\frac{\pi^3}{32}$.