Lecture 5-12: Two counterexamples and one more power series

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By way of contrast with Taylor series I now give an example of a function defined by a uniformly convergent series that is continuous everywhere but differentiable nowhere; this example is essentially the same as the one in the last section of Chapter 9 (pp. 266,7) in the text. Let $f(x) = f_0(x)$ be the distance from x to the nearest integer, so that $f(x) = x$ for $x \in [0, 1/2]$ while $f(x) = 1 - x$ for $x \in [1/2, 1]$; note that $f(x)$ is periodic with period 1. For every positive nonnegative integer i let $f_i(x) = f(4^i x)$, the distance from $4^i x$ to the nearest integer. Finally let $g(x) = \sum_{i=0}^{\infty} 4^{-i} f_i(x)$.

Since $0 \leq f_i(x) \leq 1/2$ for all x, the ith term of this sum is bounded in absolute value by $4^{-i}/2$, whence the series converges (absolutely and) uniformly to a continuous function, by the Weierstrass M-test. I claim that f is not differentiable at any $x \in \mathbb{R}$. To prove this it suffices to produce a sequence x_i converging to x such that the difference quotient $\frac{g(x)-g(x_i)}{x-x_i}$ has no limit as $i \to \infty$. To this end, use decimal notation in base 4 to write any $x \in \mathbb{R}$ as $n + y$ for some $n \in \mathbb{Z}$ and $y = 0.d_1 d_2 \ldots = \sum_{i=1}^{\infty} d_i 4^{-i}$, where each d_i is $0, 1, 2$, or 3. To avoid ambiguity, do not allow $d_i = 3$ for all but finitely many i, replacing any such expansion by the equivalent one for which $d_i = 0$ for all but finitely many i. Set $y_i = y \pm 4^{-i}$, where the sign is $+$ if $d_i = 0$ or 2, while it is $-$ if $d_i = 1$ or 3; then set $x_i = n + y_i$. Then it is easy to check that $f_j(x_i) = f_j(x) \pm 4^{j-i}$ if $j \leq i$, while $f_j(x_i) = f_j(x)$ if $j > i$.

It follows that the difference quotient $\frac{g(x)-g(x_i)}{x-x_i}$ is a sum of i terms, each ± 1, so is an even integer if i is even and an odd integer if i is odd. But no sequence alternating between even and odd integers can possibly converge (if it did, its limit L would have to be within say $1/3$ of both an even and an odd integer), whence $g'(x)$ exists for no x, as claimed. It is also not difficult to check that g is not monotone on any interval $[a, b]$ with $a < b$. Roughly speaking, the graph of g is infinitely krinkly. The kinks in the graph of each f_i, preventing it from being differentiable at more and more points as $i \to \infty$, combine to destroy the differentiability of g at any point. In fact there are other examples of infinite series $f(x) = \sum_i f_i(x)$ such that each f_i is differentiable everywhere and yet f is still differentiable nowhere.

To continue with my parade of horrors, I now present an example of a function that is infinitely differentiable everywhere but which fails to be analytic at $x = 0$ (that is, does not admit a convergent expansion in powers of x). See Theorem 8.22 on p. 221. Set $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0, f(0) = 0$. The chain rule shows that f is indeed infinitely differentiable at all $x \neq 0$, with its nth derivative $f^{(n)}$ taking the form $p_n(1/x)e^{-\frac{1}{x^2}}$ for some polynomial p_n. What about the point $x = 0$? Taking the limit of $p_n(1/x)e^{-\frac{1}{x^2}} = \frac{p_n(1/x)}{e^{1/x^2}}$ as $x \to 0$ is equivalent to taking the limit of $\frac{p_n(y)}{e^{y^2}}$ as $y \to \infty$; applying L'Hopital's Rule several times, we see that this last limit is 0 for any polynomial p_n. Hence all derivatives of f exist at 0 as well and are equal to 0. The Taylor series of f at $x = 0$ is the 0 series, which clearly does not converge to f. There are worse examples of infinitely differentiable functions g that are not analytic at any point.

There is one more important power series arising often in applications, namely the binomial series, or Newton's binomial expansion (see Theorem 8.18 on p. 217). To motivate this series, recall first the binomial theorem, which asserts that $(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m$. Note that one formula for the coefficient $\binom{n}{m}$, namely $\frac{n!}{m!(n-m)!}$, makes no sense if the exponent n is replaced by an arbitrary real number α; but an alternative formula, namely $\frac{n(n-1)\ldots(n-m+1)}{m!}$, does make sense with α in place of n.

Accordingly I define the binomial series as $1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n$. The ratio $|\frac{a_{n+1}}{a_n}|$ between the absolute values $|a_{n+1}|, |a_n|$ of the coefficients a_{n+1}, a_n of x^{n+1}, x^n in this series is then $\frac{|\alpha-n|}{n+1}$, which approaches 1 as $n \to \infty$; so this series has radius of convergence 1. It can be shown that this series also converges at $x = \pm 1$, provided that $\alpha > 0$.

To justify the series, I differentiate $f(x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n$ term by term and multiply the differentiated series by $1 + x$, which can be regarded as a power series with only two nonzero terms. Recall that I showed earlier that the product $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$ of two absolutely convergent power series $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ is $\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for all n. In the present case, I get $(1 + x)f'(x) = \alpha f(x)$, whence $(f(x)(1 + x)^{-\alpha})' = 0$ and $f(x) = c(1 + x)^{\alpha}$ for some constant c; plugging in $x = 0$, I show that $c = 1$.

Replacing x by $-x^2$ in the series for $(1+x)^\alpha$ and taking $\alpha = -1/2$, I get the series $1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})\cdots(-\frac{2n-1}{2})(-1)^n}{n!} x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)x^{2n}}{2^n n!}$ for $(1-x^2)^{-1/2}$, valid for $|x| < 1$. Integrating term by term and recalling the Inverse Function Theorem I get the series for $\arcsin x$, the inverse function of $\sin x$ (defined on $[-1,1]$ and taking values in $[0,\pi/2]$), namely $x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)x^{2n+1}}{2^n(2n+1)n!}$. As with the series for $\arctan x$, we pick up $x = \pm 1$ as two more points of convergence of this series. Plugging in these values, I get two series converging absolutely to $\pm\pi/2$, respectively. These series converge much faster than the Newton-Gregory series $4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ for π derived earlier.