Lecture 5-10: Power and Taylor series: examples

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Last time you learned that any Taylor series $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ with positive (or infinite) radius of convergence R can be integrated term by term within the radius of convergence, so that the series $g(x) = \sum_{k=0}^{\infty} \frac{a_k(x-a)^{k+1}}{k+1}$ converges for $|x-a| < R$ and $g(x) = \int_a^x f(t)\,dt$ for all x in this range. I also showed that the differentiated series $h(x) = \sum_{k=1}^{\infty} ka_k(x-a)^{k-1}$ also has radius of convergence R and its term-by-term integrated series coincides with $f(x)$, so that $f(x)$ is differentiable and $f'(x) = h(x)$ for $|x| < R$.

A very handy result due to the Norwegian mathematician Abel asserts that if a Taylor series $\sum_{n=1}^{\infty} a_n(x-a)^n$ converges at $a+R>a$, then the convergence is uniform on the entire closed interval $[a, a+R]$, so that the sum is a continuous function on that interval. The same result holds if the series converges at $a_R<a$, replacing $[a, a+R]$ by $[a-R, a]$. Thus, for example, since the series for $-\ln(1-x)$, namely $\sum_{n=0}^{\infty} \frac{xn+1}{n+1}$ has radius of convergence 1 and also converges (by the Alternating Series Test) at $x=-1$, its sum at that point must be $\lim_{x\to -1^+} \ln(1-x) = -\ln 2$.

In a similar way, the geometric series for $\frac{1}{1+x^2}$, namely $\sum_{n=0}^{\infty}(-1)^n x^{2n}$, which has radius of convergence 1, may be integrated term by term to the series $\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, which also has radius of convergence 1. This time this series also converges at both $x = \pm 1$, so its sums at those points must be $\pm\pi/4$. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4$ is called the Newton-Gregory series in the text.

Example

We have already seen that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$ to the familiar exponential function e^x. More generally, for any $a \in \mathbb{R}$, the Taylor series $\sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!}$ also converges for all x to e^x. Next consider the closely related series $s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Here one checks by the ratio test that both series have infinite radius of convergence; in applying this test to $s(x)$, for example, one should of course look at the ratio of two successive *nonzero* terms $\frac{(-1)^n x^{2n+1}}{(2n+1)!}, \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$, so that one does not divide by 0. By differentiation one gets $s'(x) = c(x), c'(x) = -s(x)$, whence by differentiation one gets that $s(x)^2 + c(x)^2$ is constantly equal to its value at 0, namely 1, since the derivative of this quantity is 0. I define $\sin x = s(x) \cos x = c(x)$. Note that $s''(x) = -s(x), c''(s) = -c(x)$. In this way I have defined the sine and cosine functions without geometry or trigonometry.

Example

The last example occurs on p. 261 of the text (Theorem 9.42). I can also work out power series expansions for functions that I cannot even completely write down. For example, starting from the geometric series $\frac{1}{1+x^3} = \sum_{n=0}^{\infty}(-1)^n x^{3n}$, valid for all $x \in (-1, 1)$, we get the series $\int_0^x \frac{1}{1+t^3}\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+1}}{3n+1}$, valid for all x in the larger interval $[-1, 1]$, even though there is no formula for this integral.

Adding the series for $\ln(1+x)$ and $-\ln(1-x)$, I get the series $2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$, which converges to $\ln\frac{1+x}{1-x}$ for $|x|<1$. As it turns out that any positive number y can be written as $\frac{1+x}{1-x}$ for some $x \in \mathbb{R}$ with $|x|<1$, I now have a convergent series which can be used to compute the natural logarithm of any positive number.

In general, given any function $f(x)$ which admits a Taylor series expansion $\sum_{n=0}^{\infty} a_n(x-a)^n$ with a positive radius of convergence, I can differentiate this series n times and plug in $x = a$ to deduce that $a_n = f^{(n)}(a)/n!$ for all n, where $f^{(n)}(a)$ denotes the nth derivative of f at a. This formula leads to a natural definition.

Definition; p. 209

Given an infinitely differentiable function $f(x)$ on an interval $(a-R, a+R)$ for some $R > 0$, its *Taylor series at* $x = a$ is the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$. This is the only Taylor series at $x - a$ that has a chance of converging to $f(x)$ for $x - a \in (-R, R)$. If it does converge to $f(x)$ on the interval $(a-R, a+R)$ (for some $R > 0$) we say that f is *analytic at* $x = a$.

Unfortunately it is in the nature of things that not all infinitely differentiable functions are analytic at all points. For example, it is known that the Taylor series at $x = a$ of an infinitely differentiable function can be an arbitrary series, so that there is no guarantee that this series will converge at any point other than $x = a$. Secondly, even if the Taylor series of a function f at $x = a$ does converge, it can converge to a function different from f. I will explore these matters in more detail later.