Lecture 5-1 Infinite series, continued

May 1, 2023

Last time I showed that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is a boundary point among p-series $\sum_{k=1}^{\infty} \frac{1}{k[}$, in that the p-series diverges for $p \leq 1$ but converges for $p > 1$. On the other hand, one *cannot* regard the harmonic series, or any other series, as lying on the overall boundary between convergence and divergence for general series with nonnegative terms, since there is no such overall boundary. More precisely, given a divergent series, if it diverges, there is a "substantially smaller" series which still diverges.

Theorem

Let $S = \sum_{i=1}^{\infty} a_i$ be a series with positive terms. If S diverges, then there is a sequence ϵ_i converging to 0 such that $\sum_{i=1}^{\infty} \epsilon_i a_i$ still diverges.

Proof.

Set $s_n = \sum^n a_i, \epsilon_n = \frac{1}{\sqrt{s_n}}$. Then the partial sum $\sum_{i=1}^{n} \epsilon_i a_i \geq \sum_{i=1}^{n} \epsilon_n a_i = \sqrt{s_n} \to \infty$ as $n \to \infty$. □

The parallel result (that if $\sum_i a_i$ converges there is a larger series that still converges) is a little harder to prove.

Theorem

If $\sum_{i=1}^{\infty} a_i$ is a convergent series with positive terms then there is a sequence M_i with $M_i \to \infty$ as $i \to \infty$ such that $\sum_i M_i a_i$ still converges.

Proof.

Let S be the sum of the given series. Then there is an index N_1 such that $\sum_{i=1}^{N_1} a_i > \frac{S}{2}$, so that $\sum_{i=N_1+1}^{\infty} a_i < \frac{S}{2}$. Similarly there is an index N_2 N_1 with $\sum_{i=N_2+1}^{\infty} a_i < \frac{S}{4}$ and for any k an index $N_k > N_{k-1}$ with $\sum_{i=N_k+1}^{\infty} a_i < \frac{S}{2^k}$. Set $M_i = 1$ for $1 \leq i \leq N_1$, $M_i = 2$ for $N_1 + 1 \leq i \leq N_2$, $M_i = 3$ for $N_2 + 1 \leq i \leq N_3$, and so on. Then $\sum_{i=1}^{\infty} M_i a_i = \sum_{i=1}^{\infty} a_i + \sum_{i=N_1+1}^{\infty} a_i + \sum_{i=N_2+1}^{\infty} a_i + \ldots$; this last sum is bounded by $S + \frac{S}{2} + \frac{S}{4} + \cdots = 2S$ and $M_i \to \infty$ as $i \to \infty$, as claimed. $\square$

Last time I also introduced the notion of absolute convergence and showed that any series that converges absolutely also converges. There are also series which converge without converging absolutely; such series are said to converge conditionally. To see an example I begin by recalling

Alternating Series Test (Theorem 9.15, p. 236)

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a series such that

- $a_k \geq a_{k+1} \geq 0$ for all k, and
- $a_k \to 0$ as $k \to \infty$.

Then $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

This was proved by showing that the sequence $s_1, s_3, \ldots$ of odd partial sums is decreasing and bounded below, while the sequence $s_2, s_4, \ldots$ of even partial sums is increasing and bounded above, whence both these sequences converge; their limits must be the same since $s_{2n+1} - s_{2n} = a_{2n+1} \to 0$ as $k \to \infty$. In particular the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges; the convergence is conditional since the harmonic series diverges.

In practice, conditional convergence is very slow convergence; one has to take many terms to get even a rough approximation to the sum of the series. For this reason I will mostly concentrate on criteria for absolute convergence in the next week or so; later I will illustrate a further paradoxical property of conditional convergence. First I however I want to give a very simple necessary condition for convergence of an arbitrary series.

nth Term Test: Proposition 9.5, p. 231

If $\sum_i a_i$ converges, then $a_i \to 0$ as $i \to \infty$; thus if $a_i \not\to 0$ as $i \to \infty$, then $\sum_i a_i$ diverges.

Indeed, if s_n is the nth partial sum, then s_n, s_{n-1} have a common limit S as $n \to \infty$, whence $a_n = s_n - s_{n-1} \to L - L = 0$ as $n \to \infty$.

Geometric series are the most useful ones for comparison purposes. A basic result is

Ratio Test: Corollary 9.21, p. 239

Let $\sum_n a_n$ be a series such that $\frac{|a_{n+1}|}{|a_n|} \to L$ as $n \to \infty$. Then $\sum_n a_n$ converges absolutely is $L < 1$ and diverges absolutely (i.e. does not converge absolutely) if $L > 1$.

Proof.

If $L < 1$ then choose $M < 1$ with $L < M$; then one has $\frac{|a_{n+1}|}{|a_n|} < M$ for $n \geq N$, say, so that by induction $|a_{N+k}| \leq M^k |a_N|$ for all $k \geq 0$. Then $\sum_{n=N}^{\infty} |a_n|$ converges by comparison with the convergent series $\sum_{k=0}^{\infty} M^k |a_N|$, whence $\sum_n |a_n|$ does also. If $L > 1$ then choose $M > 1$ with $L > M$. Then $|a_{n+1}| \geq M|a_n|$ for $n \geq N$, whence in particular $|a_{n+1}| > |a_n|$ and $a_n \not\to 0$ as $n \to \infty$. □

More generally, the proof shows that $\sum |a_n|$ converges if $\overline{\lim} \frac{|a_{n+1}|}{|a_n|} < 1$, while this series diverges if $\underline{\lim} \frac{|a_{n+1}|}{|a_n|} > 1$. Unfortunately, the case $L = 1$ provides no useful information; the harmonic series and the p-series with $p = 2$ both have $L = 1$. A large number of theorems were proved in the 19th century precisely to treat the case $L = 1$.

Example

Recall my earlier proof that $\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists and equals the sum of the convergent series $\sum_{n=0}^{\infty}\frac{1}{n!}$. A similar argument shows that the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$ (by the ratio test) and equals the convergent limit $\lim_{n\to\infty}(1+\frac{x}{n})^n$. I will later show that the sum $f(x)$ of this series satisfies the multiplication rule $f(x)f(y) = f(x+y)$.