Lecture 4-7: Sequences and the Least Upper Bound Property

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I have motivated my discussion of sequences in the course by the Least Upper Bound Property, using this property to show that every decimal expansion $0.d_1d_2\ldots$ represents a unique real number. I now want to vastly generalize this example, presenting a large family of examples of sequences that are guaranteed to converge even if their limits cannot be identified. To this end, define a sequence s_n to be *monotone* if it is either increasing or decreasing, that is, either $s_n \leq s_{n+1}$ for all n or $s_n \geq s_{n+1}$ for all n.

The Monotone Convergence Theorem (Theorem 2.25 on p. 38)

A monotone sequence s_n converges if and only if it is *bounded* in the sense that there is $b \in \mathbb{R}$ with $|s_n| \leq b$ for all indices n.

Proof.

If s_n is increasing and bounded, then the set $\{s_n\}$ of all terms of the sequence has a supremum L. Given $\epsilon > 0$, choose an index N with $a_N > L_\epsilon$; then one has $L - \epsilon < s_n \leq L$ for all $n \geq N$, so that s_n converges to L, as desired. Similarly, if s_n is decreasing and bounded, then it converges to the infimum of $\{s_n\}$. Finally, *any* convergent sequence s_n must be bounded; if its limit is L then one has $|s_n - L| < 1$ for $n \geq N$, say, whence $|s_n| < |L| + 1$ for all $n \geq N$, whence $|a_n| < M$ for all indices n, where M is the maximum of $|L| + 1$ and the first $N - 1$ absolute values $|s_i|$ of the terms with $i < N$. □

Note that the same proof works if one only has $a_n \leq a_{n+1}$, or $a_n \geq a_{n+1}$, for $n \geq N$, where N is a fixed index. It is often convenient in practice to apply this weaker hypothesis.

Corollary: Theorem 9.7, p. 232

Any series $\sum_{n=1}^{\infty} a_n$ with nonnegative terms a_n converges if and only if its partial sums are bounded.

Proof.

The hypothesis guarantees that the nth partial sum s_n of the series satisfies $s_n \leq s_{n+1}$, so that s_n converges if and only if it is bounded. □

Examples

The series $S = \sum_{i=0}^{\infty} \frac{1}{i!}$ converges (where I recall that $0! = 1$). Indeed, one has $i! \geq 2^{i-1}$ for $i \geq 1$, whence a typical partial sum $\sum_{i=0}^{m} \frac{1}{i!}$ of this series is bounded by $1 + 1 + \sum_{i=1}^{m-1} 2^{-(m-1)}$, and you already know that the set of partial sums of this last type is bounded. From this it follows that the sequence $a_n = (1 + \frac{1}{n})^n$ converges. Here $(1 + \frac{1}{n})^n = \sum_{i=0}^{n} (1 - \frac{1}{n}) \cdots (1 - \frac{i-1}{n}) \frac{1}{i!}$, where the product $(1 - \frac{1}{n}) \cdots (1 - \frac{i-1}{n})$ is taken to equal 1 if either $i = 0$ or $i = 1$. Every term in this sum becomes larger and one picks up an extra term if n is replaced by $n + 1$, whence $a_n \leq a_{n+1}$; moreover, the sum shows that the a_n are bounded since the partial sums of S are. In fact, the value of S is the same as $L = \lim_{n\to\infty} a_n$ (both are equal to e, the base of natural logarithms).

Example

The harmonic series $H = \sum_{i=1}^{\infty}(1/i)$ diverges. Writing s_{2^n} for the 2^nth partial sum of this series, one finds that $s_{2^{n+1}} = s_{2^n} + \sum_{i=2^n+1}^{2^{n+1}}(1/i) \geq \sum_{i=2^n+1}^{2^{n+1}}(1/2^{n+1}) = s_{2^n} + (1/2)$, so by induction $s_{2^n} \geq (n+1)/2$ and the partial sums of H are not bounded.

See Example 2.27 on p. 39 of the text. Note that the series $H_p = \sum_{i=1}^{\infty}(1/n^p)$ also diverges whenever $p < 1$, since each of its partial sums is larger than the corresponding partial sum of the harmonic series. I will show later that the series for H_p *converges* whenever $p > 1$.

I have shown that any bounded monotone sequence converges; but on the other hand many bounded non-monotone sequences (e.g. $a_n = (-1)^n$) fail to converge. There is however a very useful way to extract a convergent sequence out of any bounded sequence, whether or not that sequence itself converges. To do this I need a definition.

Definition (p. 43)

Given a sequence a_n I say that another sequence b_k is a *subsequence* of a_n if there is a strictly increasing sequence $n_1, n_2, \ldots$ of positive integers such that $b_k = a_{n_k}$.

Then I have

Theorem 2.32, text, p. 45

Every sequence a_n has a monotone subsequence.

Proof.

Call an index n a *peak index* if $a_m \leq a_n$ for all $m \geq n$. Then I consider two cases. If there are infinitely many peak indices, say $n_1, n_2, \ldots$ with $n_1 < n_2 < \ldots$, then the definition of peak shows that the subsequence $b_k = a_{n_k}$ is decreasing. If instead there are only finitely many peak indices, then there must be a largest one N (or set $N = 0$ if there are no peak indices at all). Set $n_1 = N + 1$; if $n_1, \ldots, n_k$ have been defined so that $a_{n_1} < \cdots < a_{n_k}$, then since $n_k > N$ there must be an index $m > n_k$ with $a_m > a_{n_k}$; set $n_{k+1} = m$. In this inductive way we have defined indices n_k for all k such that $b_k = a_{n_k}$ is an increasing subsequence of a_n. □

As an immediate corollary every bounded sequence has a convergent subsequence (Theorem 2.33 on p. 45 of the text), since a monotone subsequence of a bounded sequence is again bounded and so must converge.

We also have

Proposition 2.30, p. 44

Any subsequence of a convergent sequence a_n converges to the same limit.

Indeed, if a_n converges to L and b_k is a subsequence then for any $\epsilon > 0$ there is N with $k \geq N$ implies $|a_k - L| < \epsilon$. But then $k \geq N$ also implies that $b_k = a_{n_k}$, necessarily having an index $n_k \geq k$, also satisfies $|b_k - L| < \epsilon$, as desired.

Definition, text, p. 37

A subset S of $\mathbb{R}$ is *closed* if the limit L of any convergent sequence s_n of points in S also lies in S.

This definition of closed set agrees with the one given in the first week of class, since given any convergent sequence s_n there is a monotone subsequence t_n that converges to the same limit. Thus if a subset S of $\mathbb{R}$ is closed in the sense that it contains the supremum and infinimum of any bounded subset then it must contain the supremum or infimum of the subsequence t_n, which is the same as the limit of both t_n and s_n. Conversely, if S contains the limit of any convergent sequence of numbers in it and $B \subset S$ is bounded above, with supremum b, then for every n we can choose $s_n \in B$ with $b \geq s_n > b - (1/n)$. The sequence s_n must then converge to b, whence $b \in S$, as desired. Similarly S must also contain the infimum of any of its subsets that is bounded below.