

Lecture 4-3: Sequences of real numbers

April 3, 2023

Last week I showed that in order to make sense of an infinite sum $\sum_{i=1}^{\infty} a_i$ it is necessary to consider the *partial sums* $s_n = \sum_{i=1}^n a_i$, which form a succession $s_1, s_2, \dots$ of numbers. If this succession “crunches down” on some number s in some sense, then that number s deserves to be called the value of the sum $\sum_{i=1}^{\infty} a_i$.

But now what does it mean for succession of numbers to crunch down on another one? To answer this I first define the notion of succession formally.

Definition, p. 23, text

A sequence (of real numbers) is a function s from the natural numbers to the real numbers. I write s_n rather than $a(n)$ for the value of s at n and denote the entire sequence by $\{s_n\}$, or just s_n .

One often gives a formula for a_n , either directly as a function of n or inductively; for example, there is the sequence of reciprocals given by $a_n = 1/n$ and the famous *Fibonacci sequence* given by $s_1 = s_2 = 1, s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$. Although there is in fact a formula for s_n in terms of n in this last example, it is important to realize that such a formula is not actually necessary to define the sequence (s_n) ; for this the inductive recipe $s_n = s_{n-1} + s_{n-2}$ is enough.

Definition, p. 25, text

An *infinite series* is an expression of the form $\sum_{i=1}^{\infty} a_i$, where the a_i are real; sometimes I allow the index of summation to start at a different number than 1. I will identify any such series with its sequence of **partial sums** s_n , where $s_n = \sum_{i=1}^n a_i$.

Definition, p. 26

I say that the sequence $\{s_n\}$ *converges to the limit* L if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ for all $n \geq N$. I write $\lim s_n = \lim_{n \rightarrow \infty} s_n = L$ or $s_n \rightarrow L$ (and say s_n approaches L) as $n \rightarrow \infty$ in this situation. I can also say that $\{s_n\}$ converges without specifying its limit.

For example, the sequence with $a_n = (-1)^n/n$ converges to 0: given $\epsilon > 0$, choose $N > 1/\epsilon$ by the Archimedean Property; then $|\frac{(-1)^n}{n}| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$ for $n \geq N$, whence $|a_n| < \epsilon$ for $n \geq N$, as desired. Note that 0 is neither the supremum nor the infimum of the set $\{a_n\}$ of terms of the sequence in this case.

On the other hand, as shown in the text, the sequence $\{(-1)^n\}$ does not converge (one says it **diverges**). Indeed, suppose that this sequence converges to a . There would then have to be some N such that $|(-1)^n - a| < \frac{1}{2}$ for all $n \geq N$. If n is odd, this implies that $|-1 - a| < \frac{1}{2}$; if n is even, this implies that $|1 - a| = |a - 1| < \frac{1}{2}$. Applying the Triangle Inequality I get $|-1 - 1| = 2 < \frac{1}{2} + \frac{1}{2} = 1$, which is absurd. This is the first application of the Triangle Inequality in this course.

Often one can understand a complicated sequence by comparing it to a simpler one. Thus for example the text shows in Example 2.8 on p. 27 that $\lim_{n \rightarrow \infty} (\frac{2}{n^2} + \frac{4}{n} + 3) = 3$. To see this observe that $\frac{2}{n^2} \leq \frac{2}{n}$ for $n \geq 1$, whence $0 \leq \frac{2}{n^2} + \frac{4}{n} \leq \frac{6}{n}$ and $|\frac{2}{n^2} + \frac{4}{n} + 3 - 3| \leq \frac{6}{n}$. Given $\epsilon > 0$, if we choose $N > \frac{6}{\epsilon}$ then $\frac{6}{n} < \epsilon$ for $n \geq N$ whence the absolute value is less than ϵ , as desired.

Sometimes clever algebraic manipulations make it easy to evaluate certain limits. Thus for instance if $a_n = \sqrt{n+1} - \sqrt{n}$ then we can rewrite a_n as $\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$, whence it easily follows that $a_n \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, consider the series $\sum_{i=1}^{\infty} (\sqrt{i+1} - \sqrt{i})$. This is a so-called **telescoping series**; its n th partial sum $s_n = \sum_{i=1}^n (\sqrt{i+1} - \sqrt{i}) = \sqrt{n+1} - \sqrt{1}$. Clearly $s_n = \sqrt{n+1} - 1 \rightarrow \infty$ in the sense that for any M there is N such that $s_n > M$ for $n \geq N$. The same argument shows that even if we begin the series at a different index, say at $i = m$, then the n th partial sum $s_{n,m}$ of this series still approaches ∞ as n does.

This example has the following remarkable consequence. Given $\epsilon > 0$ choose N large enough that $a_n < \epsilon$ for $n \geq N$. The partial sums of $\sum_{i=N}^{\infty} a_i$ approach ∞ as n does, so given any positive real number r there is a smallest partial sum $s_{n,N} = \sqrt{n+1} - \sqrt{N}$ greater than r . But then $s_{n-1,N} = \sqrt{n} - \sqrt{N} < r$, whence $\sqrt{n+1} - \sqrt{N}$ lies between r and $r + \epsilon$. I conclude that the set S of differences $\sqrt{m} - \sqrt{n}$ of square roots of positive integers m, n for $m > n$ is dense in $\mathbb{R}^+$, whence by interchanging m and n we see that the set S' of differences $\sqrt{m} - \sqrt{n}$ of square roots of arbitrary positive integers is dense in $\mathbb{R}$. One might well have guessed that the set S' is discrete; but in fact, like $\mathbb{Q}$, it is dense in $\mathbb{R}$.

The Binomial Theorem implies that $(1 + r)^n \geq 1 + nr$ for any $r > 0$ and positive integer n , whence $(1 + r)^n \rightarrow \infty$ as $n \rightarrow \infty$; thus $\alpha^n \rightarrow \infty$ if $\alpha > 1$. If instead $0 < \alpha < 1$, then $(1/\alpha) > 1$, so $(1/\alpha)^n \rightarrow \infty$ and $\alpha^n \rightarrow 0$ as $n \rightarrow \infty$. I will derive consequences of this calculation for certain infinite series next time.