Lecture 4-28: Infinite series

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For the rest of the course I will turn to a systematic study of Chapter 9, studying not only infinite series of real numbers but also infinite series of functions, eventually representing functions as infinite series. A recurrent theme will be studying new series by relating them to old ones. This theme is easiest to implement for series $\sum_{n=0}^{\infty} a_n$ of nonnegative terms a_n. We already know that this series converges if and only if its partial sums are bounded.

Comparison Test: Corollary 9.8, p. 233

With notation as above, if $a_n \leq b_n$ for all n and if $\sum b_n$ converges, then so does $\sum_n a_n$; similarly, if $\sum_n a_n$ diverges, then so does $\sum_n b_n$.

Proof.

Indeed, if $a_n \leq b_n$ for all n, then any partial sum $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$. Then $\sum b_k$ converges if and only if its partial sums are bounded, and if so the partial sums of $\sum_k a_k$ are bounded (by the same number), so that $\sum_k a_k$ converges. Similarly, if instead $\sum_k a_k$ diverges, then its partial sums are unbounded, whence so are the partial sums of $\sum_k b_k$ and the latter series diverges. $\square$

Here I don't need the hypothesis $a_k \leq b_k$ for all k; it is enough if this inequality holds for all but finitely many values of k, say for $k \geq N$, for if so and $\sum_k b_k$ converges, then so too does $\sum_{k=N}^{\infty} b_k$, and then so do both $\sum_{k=N}^{\infty} a_k$ and $\sum_k a_k$. Note also that if the inequalities go the wrong way (so that $a_n \leq b_n$ for all n but $\sum b_n$ diverges), then *no information* can be drawn from the test; $\sum_n a_n$ might converge or diverge.

Example

The series $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ converges, since $\frac{1}{k2^k} \leq \frac{1}{2^k}$ for all k and $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges. The series $\sum_{k=1}^{\infty} \frac{k-9}{k^2}$ diverges, since one has $\frac{k-9}{k^2} > \frac{1}{2k}$ for sufficiently large k and $\sum_{k=1}^{\infty} \frac{1}{2k}$ diverges (it is a multiple of the harmonic series).

In practice even the condition that $a_k \leq b_k$ or $a_k \geq b_k$ for sufficiently large k is often inconvenient to verify. The following criterion often comes in handy.

Limit Comparison Test: see Exercise 8 on p. 240

Given two series $\sum a_n, \sum b_n$ with nonnegative terms, suppose that the ratio $\frac{a_k}{b_k}$ approaches a finite nonzero limit L as $k \to \infty$. Then $\sum a_k, \sum b_k$ converge or diverge together (that is, either one of these converges if and only if the other does).

Proof.

If $\frac{a_k}{b_k} \to L \neq 0$ then we have $L/2 < \frac{a_k}{b_k} < 2L$ for all sufficiently large k, whence if $\sum b_k$ converges so to does $\sum a_k$, by comparison with $\sum 2Lb_k$; similarly, if $\sum b_k$ diverges, then so too does $\sum a_k$ by comparison with $\sum (L/2) a_k$. $\square$

Example

Thus $\sum_{k=1}^{\infty} \frac{k^3+10k^2+27}{k^4+25k+1}$ diverges, by limit comparison with $\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{k^3}{k^4}$; one does not have to compute or even care exactly for which k one has $\frac{k^3+10k^2+27k}{k^4+25k+1} \geq \frac{1}{k}$.

Note that if $L = 0$ and $\sum_k b_k$ converges, then $\sum_k a_k$ does; likewise if $L = \infty$ and $\sum_k b_k$ diverges, then so does $\sum_k a_k$. But even with the Limit Comparison Test in hand, one needs a larger repertoire of series with known convergence behavior than just geometric series and the harmonic series.

Ordinarily the next test to consider would be the Integral Test. (Corollary 9.11, p. 233). While I am not averse to using integrals when I really need them, in this case I will avoid them, invoking a surprisingly powerful alternative called the Cauchy Condensation Test. Given a series $\sum a_k$ with nonnegative *decreasing* terms, so that $a_k \geq a_{k+1}$ for all k, it shows (rather surprisingly) that its convergence or divergence depends only on the terms a_{2^k} indexed by powers of 2.

Cauchy Condensation Test

With notation as above, $\sum_k a_k$ converges if and only if $\sum_k 2^k a_{2^k}$ converges.

Proof.

The hypotheses guarantee for each k that the sum $\sum_{i=2^k+1}^{2^{k+1}} a_i$ is bounded below and above by $2^k a_{2^{k+1}}$ and $2^k a_{2^k}$, respectively; since the sum $\sum_{i=2}^{\infty} a_i$ coincides with the double sum $\sum_{k=0}^{\infty} \sum_{i=2^k+1}^{2^{k+1}} a_i$, converging if and only if the double sum converges, the result follows. $\square$

Example

Now I can finally establish a boundary line between convergence and divergence for one family of series mentioned earlier. For p a positive real number, the p-series is defined to be $\sum_{k=1}^{\infty} \frac{1}{k^p}$. Since the geometric series $\sum_{k=1}^{\infty} \frac{2^k}{2^{kp}}$ converges if and only if $p > 1$, the same is true of the p-series: it converges if and only if $p > 1$ (see Corollary 9.13 on p 235). With this series in hand one can deduce that the series $\sum_{k=2}^{\infty} \frac{1}{k(\log_2 k)^p}$ converges if and only if $p > 1$, since the condensed version of this series is the p-series. (Here I am assuming and using the basic properties of the logarithm function.)

What about series that do not consist of nonnegative terms?

Definition

I say that the series $\sum_k a_k$ converges *absolutely* if $\sum_k |a_k|$ converges.

Theorem 9.18, p. 237

if $\sum_k a_k$ converges absolutely, then $\sum_k a_k$ converges.

Proof.

If $\sum_k |a_k|$ converges, then $\sum_k (a_k + |a_k|)$ has nonnegative terms, with the kth one bounded by $2|a_k|$, whence this series converges. Subtracting off the convergent series $\sum_k |a_k|$ I deduce that $\sum a_k$ converges. $\square$

Thus for example the series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges, since $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$ converges by comparison with $\sum_{k=1}^{\infty} \frac{1}{k^2}$, so the given series converges absolutely.