# Lecture 4-26: Wrapping up derivatives

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I will begin by wrapping up the example from last time, which involved the function $f(x)$ defined by $x^2 \sin(1/x)$ if $x \neq 0, f(0) = 0$. We have seen that the derivative $f'(x)$ of this function is given by $f'(x) = 2x \sin(1/x) - \cos(1/x)$ if $x \neq 0$ while $f'(0) = 0$. Now the function $g(x)$ defined by $g(x) = 2x \sin(1/x)$ if $x \neq 0, g(0) = 0$, is easily seen to be continuous, whence by the Fundamental Theorem of Calculus it is a derivative, as is the difference $h(x) = g(x) - f'(x) = \cos(1/x)$ if $x \neq 0, h(0) = 0$.

Then the function $k(x)$ defined by $(1-x)\cos(1/x)$ if $x \neq 0, k(0) = 0$, is then also a derivative, since the function $h(x) = x\cos(1/x)$ for $x \neq 0, h(0) = 0$, is continuous and so is a derivative. Note that the supremum of $k(x)$ on [0, 1] is 1 while its infimum is -1, and that it fails to attain either of these values. Thus the Extreme Value Property, unlike the Intermediate Value Property, can fail for derivatives, even on closed bounded intervals.

So can the Boundedness Property: if we take $r(x) = x^2 \sin(1/x^2)$ for $x \neq 0, r(0) = 0$, then $r'(x) = 2x \sin(1/x^2) - (2/x) \cos(1/x^2)$ for $x \neq 0$ while $r'(0) = 0$ and $r'$ is not bounded on $[0, 1]$.

If we take $F(x) = f(x) + \frac{x}{2} = \frac{x}{2} + x^2 \sin(1/x)$ for $x \neq 0$, $F(0) = 0$, then $F'(0) = 1/2$ while $F'(x) < 0$ for certain values of $x$ arbitrarily close to 0. It follows that there is no interval $[0, a]$ for $a > 0$ such that $F$ is weakly increasing on $[0, a]$, even though $F'(0) > 0$. The connection between monotonicity and the sign of the first derivative is weaker than one might expect if the first derivative is discontinuous.

Next I will state the well-known second derivative test for a local maximum or minimum:

**Theorem 4.22, p. 106**

Let $f$ be twice differentiable on an open interval $(a, b)$ and suppose that $x_0 \in (a, b)$ is such that $f'(x_0) = 0, f''(x_0) > 0$. Then $x_0$ is a local minimum of $f$. Similarly if $f'(x_0) = 0, f''(x_0) < 0$, then $x_0$ is a local maximum of $f$.

## Proof.

If $f'(x_0) = 0, f''(x_0) > 0$, then the definition of $f''(x_0)$ shows that there is an interval $(x_0 - \delta, x_0 + \delta)$ such that $f'(x) > 0$ for $x \in (x_0, x_0 + \delta)$ while $f'(x) < 0$ for $x \in (x_0 - \delta, x_0)$. The Mean Value Theorem then guarantees that $f$ is is strictly decreasing on $(x_0 - \delta, x_0]$ and strictly increasing on $[x_0, x_0 + \delta)$. This implies that $f$ has a local minimum at $x_0$, as claimed. The other case $f'(x_0) = 0, f''(x_0) < 0$ is similar. □

More generally, suppose that $f'(x_0) = 0$, $f$ is $n$ times differentiable at $x_0$, and that $n$ is the smallest positive integer with $f^{(n)}(x_0) \neq 0$. Then, if $n$ is even, we get the same criterion for $x_0$ to be a local maximum or minimum as for the case $n = 2$, while if $n$ is odd then $x_0$ is neither a local maximum or a local minimum for $f$; we call $x_0$ as saddle point for $f$ in this situation. (If, as is possible even for nonconstant $f$, we have $f^{(n)}(x_0) = 0$ for all $n$, then we are out of luck; no information can be deduced.)

There is a refinement of the Mean Value Theorem called the Cauchy (or parametric) Mean Value Theorem.

## Theorem 4.23, p. 111

Let the functions $f, g$ be continuous on $[a, b]$ and differentiable on $(a, b)$ with $g'(x) \neq 0$ on $(a, b)$. Then there is $x_0 \in (a, b)$ with $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$.

## Proof.

To begin with, Rolle's Theorem guarantees that $g(b) \neq g(a)$, so that the conclusion makes sense. Next, setting $m = \frac{f(b)-f(a)}{g(b)-g(a)}$ and applying Rolle's Theorem to $h(x) = f(x) - f(a) - m(g(x) - g(a))$, we get $h'(x_0) = 0$ for some $x_0$, since $h(a) = h(b) = 0$, and the conclusion follows immediately. □

Note that if we applied the Mean Value Theorem directly to the ratio $\frac{f(b)-f(a)}{g(b)-g(a)} = (\frac{f(b)-f(a)}{b-a})/(\frac{g(b)-g(a)}{b-a})$ we would get that this ratio equals $f'(x_0)/g'(x_1)$ for some $x_0, x_1 \in (a, b)$. Cauchy's version of this theorem implies the stronger result that we can take $x_0 = x_1$.

As an immediate consequence we get

### L'Hopital's Rule

Let $f, g$ be differentiable on an open interval $I$ and $x_0 \in I$. Suppose that $f(x_0) = g(x_0) = 0$, $g(x) \neq 0$ for $x \in I, x \neq x_0$, and that $L = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists and equals $L$.

This follows at once since for $x \neq x_0$ we have $\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(y)}{g'(y)}$ for some $y$ between $x_0$ and $x$.

Finally let me recall the well-known Leibniz notation: If $f$ is a differentiable function of $x$, then the derivative $f'(x)$ of $f$ is also often denoted $\frac{df}{dx}$, even though $\frac{df}{dx}$ is a limit of a quotient rather than an actual quotient. If $f, g$ are differentiable functions such that the composite function $f(g)$ is defined on an open interval, then $f(g)$ is differentiable there and $f(g)'(x) = f'(g(x)g'(x)$, Writing $\frac{df}{dx}$ for $f(g)'(x)$, $\frac{df}{du}$ for $f'(g(x))$ and $\frac{du}{dx}$ for $u'(x)$, so that the variable $u$ represents the function $g$, we can write the Chain Rule as $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$. The formal cancellation of $du$ makes it easy to remember this rule.