Lecture 4-24: Derivatives, continued

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Continuing with derivatives, we now show that inverses of one-to-one differentiable functions are themselves differentiable, provided that their derivatives are never 0.

Theorem 4.11, p. 97

Let f be differentiable and one-to-one on the open interval $[a, b]$ and suppose that $x_0 \in [a, b]$ is such that $f'(x_0) \neq 0$. Set $y_0 = f(x_0)$. Then the inverse function $g = f^{-1}$, defined on the interval $[c, d] = f([a, b])$ is differentiable at y_0 with $g'(y_0) = \frac{1}{f'(x_0)}$.

Proof.

We have already seen that $[c, d] = f([a, b])$ is indeed a closed interval. Then we have
$1 = \lim_{y \to y_0} \frac{f(g(y)) - f(g(y_0))}{y - y_0} = \lim_{y \to y_0} \frac{f(g(y)) - f(g(y_0))}{g(y) - g(y_0)} \frac{g(y) - g(y_0)}{y - y_0}$.
Continuity of g at y_0 forces the first fraction to approach $f'(g(y_0) = f'(x_0) \neq 0$ as $y \to y_0$, whence the second fraction approaches $\frac{1}{f'(x_0)} = g'(y_0)$ as $y \to y_0$, as claimed. □

The same argument shows that g is *not differentiable at* y_0 *if* $f'(x_0) = 0$*, since if it were we would have* $1 = 0g'(y_0)$

Example

If $f(x) = x^n$ with $n \in \mathbb{N}$, then we have already seen that $f'(x) = nx^{n-1}$ for all x; we also know that $f(x)$ has a continuous inverse $g(x) = x^{1/n}$ defined for $x \geq 0$. Evaluating $f'(g(x))$ we get $nx^{\frac{n-1}{n}}$ if $x > 0$; taking the reciprocal we get $g'(x) = \frac{1}{n}x^{\frac{1-n}{n}}$ for $x > 0$, but $g'(0)$ does not exist, as predicted by the above proof. See Example 4.13 on p. 98 of the text.

Example

Before giving a second example we recall the Fundamental Theorem of Calculus (Theorem 6.29 on p. 168 of the text), which states that any continuous function f on an interval $[a, b]$ is the derivative of its integral $F(x) = \int_a^x f(t)\,dt$. We will assume this result. In particular the function $f(x) = \int_1^x \frac{1}{t}\,dt$ is differentiable with $f'(x) = \frac{1}{x}$ for $x > 0$. It will come as no surprise to you to learn that $f(x)$ is the natural logarithm of x, denoted $\ln x$.

Since the derivative of $\ln x$ is always positive, this function is strictly increasing (as we will see shortly), so that it is one-to-one and has a differentiable inverse. This inverse is none other than your old friend $g(x) = e^x$; the Inverse Function Theorem implies that $g'(x) = g(x)$. Thus without having to assume anything about the exponential function we have shown that there is a strictly increasing function which equals its own derivative. I will later give a different and independent construction of $g(x)$, That functions with positive derivatives are strictly increasing follows at once from the Mean Value Theorem. To prove that theorem we need a result of independent interest, namely the well-known connection between derivatives and maxima or minima of functions.

Lemma 4.16, p. 102

Let the differentiable function f on the interval (a, b) have a local maximum or minimum at $x_0 \in (a, b)$. Then $f'(x_0) = 0$.

Proof.

For definiteness assume that x_0 is a maximum; the case where it is a minimum is similar. Taking limits as $x \to x_0^-$, we see that $f'(x_0) \geq 0$; taking limits as $x \to x_0^+$ we see that $f'(x_0) \leq 0$. Hence $f'(x_0) = 0$. $\square$

The same proof shows that if f is differentiable on the *closed* interval $[a, b]$ and has a local maximum at the left-hand endpoint $x = a$, then $f'(a) \leq 0$; likewise if f has a local maximum at the right-hand endpoint $x = b$, then $f'(b) \geq 0$. If instead f has a local minimum at $x = a$ or $x = b$ then these inequalities are reversed.

Rolle's Theorem, p. 103

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then we have $f'(x) = 0$ for some $x \in (a, b)$.

Proof.

Any such f must have a maximum and a minimum on $[a, b]$; if these both occur at endpoints, then f is constant and there is nothing to prove. Otherwise this result follows at once from the preceding one. $\square$

Now we can prove the Mean Value Theorem on p. 103 of the text.

Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is $x_0 \in (a, b)$ with $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Proof.

This follows from Rolle's Theorem applied to the function $g(x) = f(x) - \frac{x-a}{b-a}(f(b) - f(a))$, since $g(a) = g(b) = f(a)$. $\square$

It easily follows that

Lemma 4.19, p. 104, and Corollary 4.21, p. 105

A function f is constant on an interval (a, b) if and only if its derivative f' exists and equals 0 there. If f is differentiable on (a, b) with $f'(x) > 0$ there, then f is strictly increasing on (a, b).

Another consequence, this time that you probably have not seen before, is that any derivative satisfies the Intermediate Value Property:

Darboux's Theorem

Let f be differentiable on $[a, b]$. Then f' takes on every value c between $f'(a)$ and $f'(b)$.

Proof.

For definiteness assume that $f'(a) < c < f'(b)$; as usual the other case is similar. Set $g(x) = f(x) - cx$. Then $g'(a) < 0, g'(b) > 0$, whence by a previous remark any minimum of g can only occur at a point $x_0 \in (a, b)$. But g must have a minimum on $[a, b]$ and the result follows. $\square$

If the derivative f' of a differentiable function were always continuous (as you might intuitively expect) then the last result would follow from the Intermediate Value Theorem; but we have already seen that this is not the case. The function f defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$ was previously shown to have derivative equal to $2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$ while $f'(0) = 0$. What follows from the Intermediate Value Property of derivatives is that derivatives g' cannot have *jump* discontinuities; i.e., that one cannot have $\lim_{x \to a} g'(x)$ existing but different from $g'(a)$. Here, as mentioned last time, $\lim_{x \to 0} f'(x)$ does not exist.