Lecture 4-19: Review

April 19, 2023

This lecture will be entirely devoted to review for the midterm on Friday. I begin with the fundamental property that distinguished the real numbers from the rational ones and makes it possible to do calculus on the former, namely the Least Upper Bound Property that every nonempty set S of real numbers that is bounded above has a least upper bound (or supremum); likewise every nonempty set S of real numbers that is bounded below has a greatest lower bound (or infimum).

I constructed the real number r as a so-called cut C_r of rational numbers, more precisely taking C_r to consist of the rational numbers strictly less than r. Then $r \leq s$ if and only if $C_r \subseteq C_s$. The cut C corresponding to any nonempty set S of real numbers that is bounded above is simply the union of the cuts C_s corresponding to each element s of S, so that the Least Upper Bound Property is satisfied.

Having constructed the real numbers, I turned to sequences and series. A sequence $s = s_n$ is just a choice of real numbers s_n, one for every $n \in \mathbb{N}$. A series $\sum_{i=1}^{\infty} t_i$ is just by definition the sequence s_n of its partial sums, where $s_n = \sum_{i=1}^{n} t_i$. The sequence s_n converges to the (finite) limit L if for every $\epsilon > 0$ there is an index N such that $|s_n - L| < \epsilon$ whenever $n \geq N$; a series $\sum_{i=1}^{\infty} t_i$ thus converges to its finite sum S if and only if for every $\epsilon > 0$ there is an index N such that $|\sum_{i=1}^{n} t_i - S| < \epsilon$ whenever $n \geq N$. Be careful not to confuse a sequence (t_i) with the series $\sum_{i=1}^{\infty} t_i$ whose terms are the t_i. For example, if $t_i = 1/i$, then $t_i \to 0$ as $i \to \infty$, but the series $\sum_{i=1}^{\infty} t_i$ diverges to ∞; that is, its partial sums get arbitrarily large.

The Monotone Convergence Theorem guarantees that a monotone sequence (t_n), that is, one such that either $t_n \leq t_{n+1}$ for all n or $t_n \geq t_{n+1}$ for all n, converges if and only if it is bounded, so that there is $M \in \mathbb{R}$ with $|t_n| < M$ for all n. As an immediate consequence, a series $\sum_{i=1}^{\infty} a_i$ with $a_i \geq 0$ for all i converges if and only if its partial sums are bounded. A direct calculation shows that the geometric series $\sum_{i=0}^{\infty} r^i$ converges if and only if $|r| < 1$; its sum in this case is $\frac{1}{1-r}$. On the other hand, the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ has unbounded partial sums and accordingly diverges.

Any sequence a_n always has a monotone subsequence $b_k = a_{n_k}$ (so that the indices n_i satisfy $n_1 < n_2 < \ldots$). In particular, if a_k is bounded, so that the a_k all lie in a closed bounded interval $[a, b]$, then a_k has a subsequence converging to some $c \in [a, b]$; we express this property by saying that the closed interval $[a, b]$ is sequentially compact.

Turning now to limits of real-valued functions of a real variable, given a function f defined on an open interval $(a-d, a+d)$ for some $d > 0$ except possibly at the point a, we say that $\lim_{x \to a} f(x) = L$ if for all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x-a| < \delta$ implies that $|f(x) - L| < \epsilon$ (so that in particular $f(x)$ is defined if $0 < |x-a| < \delta$). We say that $f(x)$ is continuous at a if $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$, so that for all $\epsilon > 0$ there is $\delta > 0$ such that if $0 \leq |x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$; note that there is no need to exclude the case $x = a$ from this definition. If f is continuous at a and defined at all points of a sequence (a_n) converging to a, then $f(a) = f(\lim_{n \to \infty} a_n) = \lim_{n \to \infty} f(a_n)$, so that applying f commutes with taking the limit of a sequence converging to a.

A function defined on an open interval (a, b) is continuous (without qualification) if it is continuous at all points of (a, b). If instead f is defined on a closed interval $[a, b]$ then it is standard to weaken the definition of continuity slightly at the endpoints, saying that f is continuous at a if $\lim_{x \to a^+} = f(a)$ and similarly that f is continuous at b if $\lim_{x \to b^-} = f(b)$ (thus allowing for the possibility that f is not defined at any point to the left of a or the right of b). Then f is continuous on $[a, b]$ if and only if it is continuous at all points of this interval.

The standard limit laws apply to both sequences and functions: thus the limit of a sum, difference, or product of functions at a point is the sum, difference, or product of their limits at that point. The same holds for quotients, provided that the denominator does not have the limit 0 at the point in question. We can also take limits of functions on sequences, with one proviso: if $\lim_{n\to\infty} a_n = a$ and $\lim_{x\to a} = L$, then $\lim_{n\to\infty} f(a_n) = L$, *provided* that $a_n \neq a$ for all a. We have to impose this last condition because saying that $\lim_{x\to a} f(x) = L$ says nothing at all about $f(a)$ itself. For continuity this extra condition disappears: if f is continuous at a, $a_n \to a$ and $f(a_n)$ is defined for all n, then $f(a_n) \to f(a)$.

Similarly, we have the chain rule for continuous functions: if g is continuous at a and f is continuous at $g(a)$ then the composite function $f(g)$ is continuous at a.

Recall now the three three key properties of continuous functions:

- Boundedness Property: any continuous function on a closed bounded interval $[a, b]$ is bounded.
- Extreme Value Property: any such function on $[a, b]$ has both a maximum and a minimum there (not just a supremum and infimum).
- Intermediate Value Property; any such function f on $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

The last of these properties is often applied to differences $f - g$ of continuous functions rather than to f and g separately. All three properties, taken together, say that the image $f[a, b]$ of a closed bounded interval $[a, b]$ under a continuous function f is another closed bounded interval $[c, d]$.

I will conclude with some remarks about how functions can fail to be continuous. First of all, it is easy for a function f on an interval $[a, b]$ *not* to be continuous at *any* point of $[a, b]$: just take $f(x) = 0$ if $x \in \mathbb{Q}$, $f(x) = 1$ if $x \notin \mathbb{Q}$. More interestingly, it is possible for f to be discontinuous at all rational x yet continuous at all irrational x: define f on $[1, 2]$ by decreeing that $f(x) = 0$ if $x \notin \mathbb{Q}$ while $f(x) = 1/n$ if $x = m/n \in \mathbb{Q}$ in lowest terms, with $m, n > 0$. This is Exercise 9 on p. 74 in the study problems, which I urge you to think about carefully. Interestingly, the opposite phenomenon is impossible: there is no function continuous at all $x \in \mathbb{Q}$ but discontinuous at all $x \notin \mathbb{Q}$.

Finally, some logistics: you will do all your work on the test paper and are permitted one sheet (front and back) of handwritten notes.