Lecture 4-17: Derivatives

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Just one more property of continuous functions to discuss before we get to derivatives, the topic of Chapter 4 in the text.

Definition, p. 66, text

We say that the function f defined on a set S is *uniformly continuous on* S if for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $x, y \in S$ and $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

The key difference between this property and ordinary continuity is that, given $\epsilon > 0$ the *same* δ has to work for all $x, y \in S$; we are not allowed to use different δ for different x. For example, the function $f(x) = 1/x$ is continuous but not uniformly continuous on the open interval $(0, 1)$: taking $\epsilon = 1$, we have $f(1/n) = n, f(1/2n) = 2n$ for all $n \in \mathbb{N}$. If any $\delta > 0$ satisfied the above definition for this value of ϵ, then we could choose n with $(1/2n) < \delta$, and then the values $x = 1/n, y = 1/2n$ would contradict this definition. This is Example 3.16 on p. 67 of the text.

Once again it turns out that restricting to functions defined on closed bounded intervals fixes the problem.

Theorem 3.17, p. 68

Any function f defined and continuous on a closed bounded interval $[a, b]$ is uniformly continuous there.

Proof.

Given $\epsilon > 0$, I will show that the choice $\delta = 1/n$ must satisfy the definition for some choice of $n \in \mathbb{N}$. Otherwise for each such n we would have $x, n, y_n \in [a, b]$ with $|x_n - y_n| < 1/n, |f(x_n) - f(y_n)| > \epsilon$. Then some subsequence x_{n_k} of x_n would converge to some $x \in [a, b]$ and the condition $|x_n - y_n| < 1/n$ forces the corresponding subsequence y_{n_k} of y_n to converge to the same number x. But then $f(x_{n_k}, f(y_{n_k})$ converge to the same limit $f(x)$, contradicting $|f(x_n) - f(y_n)| > \epsilon$ for all n. □

We won't actually have any occasion to use uniform continuity in this course until much later, but it is needed in the theory of integration, covered in Chapter 6 of the text, and is useful in proving some advanced theorems in analysis. The notion of uniformity *will* occur in the context of sequences and series and will play a very important role in Chapter 9.

Now we turn to the basic definition of Chapter 4.

Definition, p. 88

Given a function f defined on an open interval (a, b) and $x_0 \in (a, b)$ we say that f is differentiable at x_0 if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. The value of this limit is denoted $f'(x_0)$. If f is defined on a closed interval $[a, b]$ then we say that f is right differentiable at $x = a$ if the one-sided limit $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists; similarly f is left differentiable at $x = b$ if $\lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}$ exists.

If f is differentiable at x_0 then it is also continuous there (Proposition 4.5, p. 91), since then $\lim_{x\to x_0}(f(x) - f(x_0)) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x - x_0) = f'(x_0)\cdot 0 = 0$. A simple application of the difference of powers formula $x^n - x_0^n = (x - x_0)(x^{n-1} + x^{n-2}x_0 + \ldots + x_0^{n-1})$ on p. 90 shows that the power function $f(x) = x^n$ is differentiable everywhere and has derivative $f'(x) = nx^{n-1}$, if $n \in \mathbb{N}$. In fact a slightly more elaborate argument shows for any $r \in \mathbb{R}$ that if $f(x) = x^r$ for $x > 0$ then $f'(x) = rx^{r-1}$ for $x > 0$.

On the other hand, the function $g(x) = |x|$ fails to be differentiable at $x = 0$, since the left-hand limit $\lim_{x\to 0^-} \frac{|x|}{x} = -1$ fails to coincide with the right-hand limit $\lim_{x\to 0^+} \frac{|x|}{x} = 1$.

The limit laws for addition and subtraction show at once that the function $f + g$ is differentiable at any point x_0 if f, g are differentiable there and $(f+g)'(x_0) = f'(x_0) + g'(x_0), (f-g)'(x_0) = f'(x_0) - g'(x_0)$. The calculation $\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x)}{x-x_0} + \frac{f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$ shows that fg is differentiable at x_0 whenever f and g are and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. Similarly we have the quotient rule that $\frac{n}{d}$ is differentiable at x_0 whenever n, d are and $d(x_0) \neq 0$; then $(\frac{n}{d})'(x_0) = \frac{d(x_0)n'(x_0)-n(x_0)d'(x_0)}{d(x_0)^2}$.

The Chain Rule, stated on p. 99 of the text, is a bit trickier. Let f be differentiable at $g(x_0)$ and g be differentiable at x_0 (so that f is defined in some open interval containing $g(x_0)$ while g is defined in some open interval containing x_0). Then the composite function $f(g)$ is differentiable at x_0 and $f(g)'(x_0) = f'(g(x_0))g'(x_0)$. To prove this we must study the limit $\lim_{x \to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0}$. There are two cases. If $g'(x_0) \neq 0$ then we must have $g(x) \neq g(x_0)$ for all x in some interval $(x_0 - a, x_0 + a)$ about x_0. For such x we may then write $\frac{f(g(x))-f(g(x_0))}{x-x_0} = \frac{f(g(x)-f(g(x_0))}{g(x)-g(x_0)} \frac{g(x)-g(x_0)}{x-x_0}$ for $x \in (x_0 - a, x_0 + a)$ and the result follows at once by taking limits.

If $g'(x_0) = 0$ then given $\epsilon > 0$ there is $\delta > 0$ such that we still have $|\frac{f(g(x))-f(g(x_0))}{x-x_0} - f'(g(x_0)g'(x_0)| = |\frac{f(g(x))-f(g(x_0))}{x-x_0}| < \epsilon$ whenever $|x - x_0| < \delta$ and $g(x) \neq g(x_0)$, using the continuity of f at $g(x_0)$ and g at x_0; but if $g(x) = g(x_0)$, then trivially $\frac{f(g(x))-f(g(x_0))}{x-x_0} = 0$ for $x \neq x_0$. Thus $f(g)'(x_0) = 0 = f'(g(x_0))g'(x_0)$ in this case too.

Example

Set $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$. Combining the product and chain rules, we get that $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$, while a direct calculation using the definition of limit shows that $f'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = 0$, since $\frac{x^2}{x} = x$ has the limit 0 as $x \to 0$, while $\sin(1/x)$ is bounded between 1 and -1 for all x (we are applying the squeeze limit law here). This example is interesting because f' always exists but is discontinuous at 0; since $\cos(1/x)$ has no limit as $x \to 0$, the limit $\lim_{x \to 0} f'(x)$ does not even exist. We will return to this example later.