Lecture 4-14: Properties of continuous functions and limits

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Last time I showed that a continuous function on a closed bounded interval $[a, b]$ takes on a minimum and maximum value there; now I will show it takes on every value in between as well.

Intermediate Value Theorem (3.11 in the text, p. 62)

Let f be continuous on $[a, b]$ and let c be a number between $f(a)$ and $f(b)$. Then there is $y \in [a, b]$ with $f(y) = c$.

Proof.

For definiteness assume that $f(a) < c < f(b)$; the other case is similar. Define two sequences $(a_n), (b_n)$ inductively, as follows. Set $a_0 = a, b_0 = b$, so that $f(a_0) < c < f(b_0)$ and $b_0 - a_0 = b - a$. If $a_n, b_n \in [a, b]$ have been defined so that $f(a_n) \leq c \leq (b_n)$ and $b_n - a_n = \frac{b-a}{2^n}$, then let $d = \frac{a_n+b_n}{2}$. If $f(d) \geq c$, then set $a_{n+1} = a_n, b_n = d$; if $f(d) < c$, then set $a_{n+1} = d, b_{n+1} = b_n$. One easily checks that $f(a_{n+1}) \leq c \leq f(b_{n+1}$ and $b_{n+1} - a_{n+1} = \frac{b-a}{2^{n+1}}$, as desired. Now the sequence a_n is increasing .and bounded above; likewise b_n is decreasing and bounded below. Letting y, z be the respective limits of a_n, b_n, we have $y = z$ since $b_n - a_n \to 0$ as $n \to \infty$. Then $f(y)$ is the common limit of the $f(a_n)$ and $f(b_n)$, whence it must be both at least and at most c. Hence $f(y) = f(z) = c$, as desired. □

As an immediate corollary I get

Theorem 3.14, text, p. 65

If f is continuous on $[a, b]$, then the range $\{f(x) : x \in [a, b]\}$ of f is another closed bounded interval $[m, M]$, where m, M are the minimum and maximum of f on $[a, b]$, respectively.

Example

Now I can massively generalize the result of an earlier homework problem. Given any positive integer n and $\alpha > 0$ I know that $(1+\alpha)^n \geq n\alpha \geq \alpha$, whence the continuous function $f(x) = x^n$ takes values less than α and other values greater than α, whence it must take the value α as well. Then f is an increasing function, so there is in fact a *unique* $y \in \mathbb{R}^+$ with $y^n = \alpha$; of course we call y the nth root of α and denote it by $\alpha^{\frac{1}{n}}$. Similarly, given any rational number $\frac{m}{n}$ and $\alpha > 0$ the power $\alpha^{\frac{m}{n}} = (\alpha^{\frac{1}{n}})^m$ is well defined. Finally, if $\alpha > 1$ and $r \in \mathbb{R}$ then I define α^r to be the supremum of all powers α^q, where q runs through the rational numbers less than r. If $0 < \alpha < 1$ then α^r is defined to be the reciprocal $\frac{1}{\beta^r}$, where $\beta = \frac{1}{\alpha}$.

This last example can itself be massively generalized.

Theorem

If f is continuous and strictly increasing on $[0, \infty)$ such that $f(x) \to \infty$ as $x \to \infty$, then given any $c \geq f(0)$ there is a unique $y \geq 0$ with $f(y) = c$.

The corresponding result also holds for strictly decreasing functions on $[0, \infty)$ such that $f(x) \to -\infty$ as $x \to \infty$. Moreover one also has

Theorem

Any continuous one-to-one function f on a closed bounded interval must be either strictly increasing or strictly decreasing on that interval.

Proof.

Otherwise there are x, y, z in the interval with $x < y < z$ and either $f(x) < f(y) > f(z)$ or $f(x) > f(y) < f(z)$. In either case there is some value c between $f(x)$ and $f(y)$ such that f takes the value c twice, once between x and y, and then again between y and z. □

Using this last result I can construct new continuous functions from old ones.

Inverse Function Theorem for continuity (Theorem 3.29, p. 78)

Let f be continuous and one-to-one on a closed bounded interval $[a, b]$, with range another closed interval $[c, d]$. Then the inverse function g of f, sending $f(x)$ back to x for all $x \in [a, b]$, is continuous from $[c, d]$ to $[a, b]$.

Proof.

Let $x \in [a, b]$ and assume for convenience assume that $a < x < b$; the cases $x = a$ and $x = b$ are similar and easier. I will show that g is continuous at $y = f(x)$. Given $\epsilon > 0$ small enough that $x - \epsilon, x + \epsilon$ both lie in $[a, b]$, let $y_1 = f(x - \epsilon), y = f(x), y_2 = f(x + \epsilon)$. Since f is strictly monotonic between $x - \epsilon$ and $x + \epsilon$ it follows that if $\delta = \min(|y_2 - f(x)|, |f(x) - y_1|)$ then any z with $|z - y)| < \delta$ is $f(w)$ for a unique $w \in (x - \epsilon, x + \epsilon)$, whence $|g(z) - x| < \epsilon$ if $|z - y| < \delta$, forcing g to be continuous at y, as desired. $\square$

In particular for any $r \in \mathbb{Q}$ the power function $f_r(x) = x^r$ is continuous on $(0, \infty)$, since the nth root function is continuous on this interval for any $n \in \mathbb{N}$ and f_r is the composite of this function and a power function with integral exponent. This function is also continuous at $x = 0$ whenever it is defined there.

I can also use the Intermediate Value Theorem to show that some rather exotic equations have solutions (though I cannot construct the solutions explicitly).

Example

The equation $\cos x = x^2$ has a solution lying in the open interval $(0, 1)$. This follows since $f(x) = \cos x - x^2$ is continuous on $[0, 1]$, takes a positive value at 0, and a negative one at 1; hence it takes the value 0 at some point of $(0, 1)$.

I now shift gears, defining limits of functions in more general contexts. First, given a function f defined on an infinite open interval (a, ∞) I define $\lim_{x \to \infty} f(x)$ exactly as for sequences, declaring that $\lim_{x \to \infty} f(x) = L$ if for every $\epsilon > 0$ there is $M \in \mathbb{R}$ with $|f(x) - L| < \epsilon$ for $x > M$, and that $f(x) \to \infty$ as $x \to \infty$ if for every $N \in \mathbb{R}$ there is $M \in \mathbb{R}$ with $f(x) > N$ for $x > M$. The arithmetic and squeeze limit laws then carry over to such limits; for example, since $1/x, -1/x \to 0$ as $x \to \infty$ I have $(\sin x/x) \to 0$ as $x \to \infty$, since $\sin x/x$ is trapped between $-1/x$ and $1/x$. Any ratio $p(x)/q(x)$ of polynomials approaches the same value as $x \to \infty$ as the ratio $\frac{a_n x^n}{b_m x^m}$ of the leading terms $a_n x^n, b_m x^m$ of $p(x), q(x)$ respectively, since the quotients $p(x)/x^n, q_x/x^m$ approach the coefficients a_n, b_m of x^n, x^m in $p(x), q(x)$, as $x \to \infty$.

If a function f defined and continuous at all points of an open interval $(a-d, a+d)$ except for a itself has a limit L as $x \to a$, then there is a unique way to define $f(a)$ so as to make f continuous on the entire interval, namely by taking $f(a) = L$. On the other hand, if a function f fails to have a limit as $x \to a$, then there is no way to define f at a so as to make f continuous there. This occurs, for example, with the function $f(x) = \sin(1/x)$, say on the interval $(-1, 1)$. Here $f(\frac{2}{(4n+1)\pi}) = 1, f(\frac{2}{(4n+3)\pi}) = -1$ for all $n \in \mathbb{N}$, whence $\lim_{x\to 0} f(x)$ does not exist (applying the sequence definition of limit, for example). We cannot define this f at 0 to make it continuous there. This example will pop up several more times later in the course.

In the next chapter we will define derivatives of differentiable functions; it turns out that any continuous function is the derivative of another one, but the converse is false: derivatives of functions need not be continuous. We will however show that any derivative satisfies one of the key properties of continuous functions described above, namely the Intermediate Value Property. Both the Extreme Value Property and the Boundedness Property can fail for derivatives, even on closed bounded intervals.