Lecture 4-12: Limits of functions and continuity

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Now I can finally start to talk about real-valued functions of a real variable, the bread and butter of calculus. Let f be such a function and $(a-d, a+d)$ an open interval about $a \in \mathbb{R}$ such that f is defined on all points of this interval with the possible exception of a itself. The basic definition is

Definition of limit of function; compare with p. 82 in the text

I say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $0 < |x-a| < \delta$ I have $|f(x) - L| < \epsilon$. I call L the limit of f at a and I write $f(x) \to L$ as $x \to a$.

The definition is formally quite similar to that of a limit of a sequence, but there are several special features:

- the real parameter δ replaces the index parameter N;
- the limit takes place at $a \in \mathbb{R}$, rather than at infinity as for a sequence;
- it does not matter what $f(a)$ is, or even whether it is defined, thanks to the condition $0 < |x - a| < \delta$ in the definition.

Example

Let $f(x) = \frac{x^2-1}{x-1}$, so that $f(x) = x + 1$ if $x \neq 1$, while $f(1)$ is undefined. Here $\lim_{x\to 1} f(x) = 2$, since for every $\epsilon > 0$ I can take $\delta = \epsilon$, and then $0 < |x - 1| < \delta$ implies that $|f(x) - 2| = |x - 1| < \epsilon$, as desired. It does not matter that f is undefined at $x = 1$. This calculation turns out to show that the function $f(x) = x^2$ is differentiable at $x = 1$ and its derivative there is 2. This is Example 3.34 in the text (p. 82).

The most important special case of the limit of a function f at a point a occurs when this limit equals the value $f(a)$ of f at a (so that in particular $f(a)$ is defined).

Definition of continuity at a point (p. 53, text)

We say that the function f defined in some open interval $(a-d, a+d)$ is *continuous at a* if $\lim_{x \to a} f(x) = f(a)$, or equivalently if for every $\epsilon > 0$ there is $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ (note that the hypothesis is $|x-a| < \delta$ rather than $0 < |x-a| < \delta$ in this case). We say that f is *continuous* if it is continuous at all points in its domain.

The limit laws for sequences carry over to functions and the proofs are the same apart from minor changes in notation. Thus if $f(x) \to L, g(x) \to M$, as $x \to a$, then one has $f(x) + g(x) \to L + M, f(x) - g(x) \to L - M, f(x)g(x) \to LM$, as $x \to a$. Also $f(x)/g(x) \to L/M$ as $x \to a$, provided that $M \neq 0$.

The above definitions of continuity and limit of a function at a point, called the $\epsilon - \delta$ criteria on p. 70 of the text, are the standard ones, used in most texts. This text defines the limit of a function at a point on p. 82 using sequences: given f defined on an interval $(a - d, a + d)$ for some $d > 0$ except possibly at the point a, one says that $\lim_{x \to a} f(x) = L$ if given any sequence a_n converging to a with $a_n \neq a$ and $f(a_n)$ defined for all n, the sequence $f(a_n)$ converges to L. Nevertheless, the two definitions are equivalent.

Theorem

We have $\lim_{x \to a} f(x) = L$ by the $\epsilon - \delta$ definition if and only if $\lim_{x \to a} f(x) = L$ by the sequence definition. In particular, f is continuous at a by the $\epsilon - \delta$ definition.

Proof.

If the $\epsilon - \delta$ definition holds and a_n is a sequence with $f(a_n)$ defined for all n, $a_n \neq a$, and $a_n \to a$, then let $\epsilon > 0$. Then there is an index N such that $n \geq N$ implies $0 < |a_n - a| < \delta$, which in turn implies that $|f(a_n) - L| < \epsilon$, forcing $f(a_n) \to L$. Conversely, if the sequence definition holds, then suppose for a contradiction that for some given $\epsilon > 0$ and any nonnegative integer n, the choice $\delta = 1/n$ never satisfies the $\epsilon - \delta$ definition, so that there is a_n with $0 < a_n - a < 1/n$, $f(a_n)$ is defined, and $|f(a_n) - f(a)| > \epsilon$. Then clearly $a_n \to a$ as $n \to \infty$ but $f(a_n) \not\to L$, contradicting the sequence definition. □

There are two extensions of the definition of a limit of a function at a point which arise often enough to be worth mentioning explicitly. Given a function f defined on an open interval $(a, a + d)$ for some $d > 0$ (but not necessarily at any point to the left of a), suppose that for any $\epsilon > 0$ there is $\delta > 0$ such that $0 < x - a < \delta$ implies $|f(x) - L| < \epsilon$. Then we write $\lim_{x \to a^+} f(x) = L$ and say that f has the right-hand limit of L as x approaches a (from above). We define the condition $\lim_{x \to a^-} f(x) = L$ similarly, if f is defined on some open interval $(a - d, a)$, but not necessarily at a or any point to the right of it, saying that $\lim_{x \to a^-} = L$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $0 < a - x < \delta$ implies that $|f(x) - L| < \epsilon$ and calling L the left-hand limit of f at a. Clearly f has the limit L at a if and only if it has L as both left- and right-hand limit at a.

There is a basic property of continuous functions most easily proved using the $\epsilon - \delta$ definition; this is

Continuity of (or chain rule for) composite functions: Theorem 3.6, p. 56

If $f(x)$ is continuous at a and $g(x)$ is continuous at $f(a)$, then the composite function $g(f(x))$ is continuous at a.

Proof.

Given $\epsilon > 0$ choose $\delta' > 0$ with $|g(y) - g(f(a))| < \epsilon$ whenever $|y - f(a)| < \delta_1$. Then choose $\delta > 0$ with $|f(x) - f(a)| < \delta_1$ whenever $|x - a| < \delta$. Then $|x - a| < \delta$ implies $|g(f(x) - g(f(a))| < \epsilon$, as desired. □

Reverting to the sequence definition I can show:

Extreme Value Theorem, p. 60

A continuous function f on a closed bounded interval $[a, b]$ has a maximum and minimum value.

Thus any max-min problem from first-year calculus must have a solution.

Proof.

I will first show that any such f is bounded above on $[a, b]$. Otherwise for every $n \in \mathbb{N}$ there would be $x_n \in [a, b]$ with $f(x_n) > n$. The sequence x_n would have a convergent monotone subsequence y_k, where $y_k = x_{n_k}$ and $n_k \geq k$, say with $y_n \to c \in [a, b]$; but it is clear that the sequence $(f(y_n))$ cannot converge, violating the sequence definition. Similarly f must be bounded below on $[a, b]$. Now let M be the least upper bound of the set of values $f(x)$ as x runs over $[a, b]$. For each n choose $x_n \in [a, b]$ with $f(x_n) > M - \frac{1}{n}$. As before some subsequence y_k of points in $[a, b]$ converges to some $c \in [a, b]$, and now it is clear that $f(c) = \lim_{k\to\infty} f(y_k) = M$ (again because if $y_k = x_{n_k}$, then $n_k \geq k$). Similarly f must take on the infimum of the set of its values on $[a, b]$, which is also its minimum. □