# Lecture 3-29: The real numbers, continued

March 29, 2023

I will begin by sketching the construction of the real numbers, as given in a number of books (but not the Fitzpatrick text), as well as a number of online sources. This will give a very clear indication of how one uses the set of rational numbers itself to plug up its own holes. The basic idea is to construct any real number $r$ as the set of $C_r$ of rational numbers $q$ with $q < r$, being careful *not* to consider at the same time the set of rational numbers $q$ with $q \leq r$, as then these two possibly different sets of rational numbers would then have to denote the same real number.

More precisely, I call a set $S$ of rational numbers a cut if it is nonempty, bounded above, has no greatest element, and contains any rational number $x < y$ whenever it contains $y$. Then the real numbers (by definition) are exactly the cuts. Given cutes $C_x, C_y$ defining the respective numbers $x, y$, the condition for $x$ to be less than or equal to $y$ is clearly the set-theoretic condition that $C_x \subseteq C_y$. Then the least upper bound of a nonempty set $\{C_i : i \in I\}$ of cuts that is bounded above (so that there is $r \in \mathbb{Q}$ with $r \notin C_i$ for any $i$) is just the union $C$ of all the $C_i$, which clearly satisfies the definition of cut. This very simple definition thus yields the Least Upper Bound Property as a consequence.

Next I have to define the arithmetic operations on these cuts so as to satisfy the axioms of a field. For addition this is straightforward: given two cuts $C, D$, define their sum $C + D$ to consist of all sums $c + d$ as $c$ runs over $C$ and $d$ runs over $D$. Taking negatives is already a little tricky; since $x < y$ if and only if $-y < -x$, we define the negative $-C$ of a cut $C$ by first taking all $-d$ as $d$ runs over the rational numbers *not* in $C$, and then removing the largest element if there is one. Thus the cut defining $-1$ consists of all rational numbers $r$ with $r \leq -1$, with $-1$ removed, so that in the end it consists exactly of the rationals $r$ with $r < -1$. By contrast, the cut defining $-\sqrt{2}$ consists exactly of the negatives $-r$ of all rational numbers $r > \sqrt{2}$; since $-\sqrt{2}$ is not rational, this set has no largest number, so no number needs to be removed from it to make it into a cut. The difference $C - D$ of two cuts $C, D$ is just the sum $C + (-D)$.

Multiplication is even trickier: the problem is that it is *not* true that if $x < y$ and $z < w$, then $xy < zw$, though this *is* true if $x, y, z, w$ are all positive. We there for start by defining a cut $C$ to be *positive* if $0 \in C$. Then the product $CD$ of two positive cuts $C, D$ consists of all product $cd$ of positive rational numbers $c, d$, lying in $C, D$, respectively, together with all rational numbers $r \leq 0$. We multiply negative cuts via the rules
$(-C)D = C(-D) = -(CD), (-C)(-D) = CD$. The multiplicative inverse $C^{-1}$ of a positive cut $C$ then consists of all $d^{-1}$ as $d$ runs over the rational numbers not in $C$, with the largest number removed if it has one, together with all rational $e \leq 0$. We extend multiplicative inverses to negative cuts by decreeing that $(-C)^{-1} = -C^{-1}$. If $C_0 = \{r \in \mathbb{Q} : r < 0\}$ is the cut defining the real number 0, then $C_0^{-1}$ is not defined. Then one can check that the set of real numbers satisfies all the properties of an ordered field.

We also deduce that every nonempty set $S$ of real numbers that is bounded below in the sense that there is $x \in \mathbb{R}$ with $x \leq y$ for all $y \in S$ has a greatest lower bound $z$, so that $z$ is the unique largest lower bound for $S$. We also call $z$ the infimum of $S$ and denote it by $\inf S$. To see that $\inf S$ exists if $S$ is nonempty and bounded below just note that $-S = \{-x < x \in S\}$ is bounded above whenever $S$ is bounded below; then $\inf S = -\sup(-S)$.

The least upper bound property immediately implies that

**The Archimedean Property**

The set $\mathbb{N}$ of positive integers is not bounded above; equivalently, given $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ with $x < n$.

**Proof.**

Indeed, if $\mathbb{N}$ were bounded above, then it would have a least upper bound $x$, whence $x - 1$ is not an upper bound for $\mathbb{N}$ and there is $n \in \mathbb{N}$ with $n > x - 1$. But then $n + 1 \in \mathbb{N}$ and $n + 1 > x$, a contradiction. □

As the Fitzpatrick text points out, this property was actually proved centuries before Archimedes, as Archimedes himself acknowledged; but it is still traditional to call this the Archimedean property. Note that an equivalent formulation of this property states that given any positive real numbers $a, b$ we have $na > b$ for some $n \in \mathbb{N}$; to see this just choose $n \in \mathbb{N}$ with $n > \frac{b}{a}$.

An important consequence of the construction of the real numbers from the rational numbers is

## The Density Theorem (Theorem 1.9, p. 15, Fitzpatrick)

For any real numbers $x, y$ with $x < y$ there is a rational number $z$ with $x < z < y$.

## Proof.

The proof is much easier with the construction of the real numbers available than it is in Fitzpatrick (who only *assumes* that the real numbers exist and satisfy the Least Upper Bound Property). Indeed, if $x < y$, then since $y$ is the least upper bound of the set $C_y$, it follows that $x$ is not an upper bound of this set, so that we can find a rational $z \in C_y$ with $x < z$. Then $x < z < y$, as desired. □

An equivalent formulation of the Density Theorem says that given any $x \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$, there is $y \in \mathbb{Q}$ with $|x - y| < \epsilon$; in words, any real number can be approximated arbitrarily closely by rational numbers.

The corresponding property of $\mathbb{Z}$ within $\mathbb{R}$ is

**Theorem 1.8, p. 14, Fitzpatrick**

For any $c \in \mathbb{R}$ there is exactly one integer $k$ in the half-open interval $[c, c+1)$.

I will refer to the text for the proof, which is straightforward.

In the remaining time I will give a heads up about problem 17 in this week's homework, which asks you to show that every positive real number $c$ has a square root. You *cannot* use the notation $\sqrt{c}$ in the proof, as the whole point is to *show* that the number $\sqrt{c}$ exists. Instead, following the hint in the text, show that if $b^2 < c$, then for suitably small $r > 0$ we have $(b + r)^2 < c$, so that $b$ cannot be an upper bound for the set $S$ of all real numbers $x$ with $x^2 < c$. Thus the least upper bound $y$ of this set cannot be such that $y^2 < c$; similarly we cannot have $y^2 > c$. Hence $y^2 = c$, as desired.