Lecture 6-5

We conclude with the singular value and polar decompositions of rectangular matrices. Let A be an $m \times n$ matrix (not assumed square) over $\mathbb{R}$; everything we say generalizes to matrices over $\mathbb{C}$ as well if transposes are replaced throughout by conjugate transposes. Set $B = A^T A$; then B is an $n \times n$ symmetric positive semidefinite matrix and as such has a unique positive semidefinite square root P. If P is positive definite (or equivalently invertible), we set $U = AP^{-1}$, so that $A = UP$; then U is an $m \times n$ isometry from the column space of P to that of A, so that $Uv \cdot Uv = v \cdot v$ if v lies in the column space of P. If A is square, then we may take U to be an orthogonal matrix in the usual sense, so that $U^T = U^{-1}$, but U is not unique in general; if in addition A is invertible then U is unique. The decomposition $A = UP$ is called the polar decomposition of A; even though the matrix U need not be unique, the matrix P is (as the only positive semidefinite square root of B). There is another version of the polar decomposition in which A is written as PU rather than UP, again with U an isometry and P positive semidefinite; in this version P is the square root of $B' = AA^T$ rather than $A^T A$.

The singular value decomposition of A factors it in a different way; we have $A = U\Sigma V$, where U, V are orthogonal matrices of sizes $m \times m$ and $n \times n$, respectively, while Σ is an $m \times n$ "diagonal" matrix (so that its ijth entry is 0 unless $i = j$; Σ is an honest diagonal matrix if and only if A is square). The entries $d_1, \ldots, d_m$ along the "diagonal" of Σ are the square roots of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of B, or equivalently of $B' = AA^T$, which is always similar to B. They are called the singular values of A and are always nonnegative. The columns of V are orthogonal eigenvectors of P, arranged so that the ith diagonal entry of Σ is the eigenvalue corresponding to the ith column of P. The singular values of A are *not* the same as its eigenvalues, even if A is square, unless A happens to be symmetric and positive semidefinite. Like the matrix P in the polar decomposition, the matrix Σ is uniquely determined by A (up to permuting its diagonal entries), but the matrices U, V are generally not unique.

We can relate the singular value decomposition of A to the geometry of the linear transformation f with matrix A (relative to the standard bases). This transformation takes the unit ball in $\mathbb{R}^n$ to an ellipsoid of dimension $r \leq m$ in $\mathbb{R}^m$, where r is the rank of A; more precisely, this ellipsoid has the equation $(y_1/\lambda_1)^2 + \ldots + (y_r/\lambda_r)^2 \leq 1$, where $y_1, \ldots, y_r$ are the coefficients of a vector v in C, the r-dimensional column space of A in $\mathbb{R}^m$, with respect to an orthonormal basis of C, and $\lambda_1, \ldots, \lambda_r$ are the nonzero singular values of A. In particular the largest singular value, say λ_1 of A, is the maximum value of $||Av||$, the length of the vector Av, as v runs through all unit vectors in $\mathbb{R}^n$; similarly the smallest nonzero singular value of A, say λ_r is the minimum nonzero value of $||Av||$ for v a unit vector in $\mathbb{R}^n$. To get the intermediate singular values $\lambda_2, \ldots, \lambda_{r-1}$, one does a "minmax" construction outlined in LADW: for all indices i between 1 and r, one looks at all subspaces R_i of the row space R of A of dimension i, minimizes $||Av||$ as v runs through unit vectors in R_i, and then chooses the subspace R_i so that the minimum value m_i of $||Av||$ is as large as possible.

For the final exam, you should be able to compute the singular values of a matrix, but I will not ask you to work out the full singular value decomposition. To wrap up the course I will mention that it turns out that any square matrix A, real or complex, is similar to its transpose A^T, but *not* to its conjugate transpose $\bar{A}^T$ if A is complex (as one sees already for 1×1 matrices. I remind you that the final will take place on Wednesday, June 10, from about 2:25 to 4:30, and should be submitted through Canvas if possible.

Let me just say once again how amazed and grateful I am at how well all of you have borne up under very difficult circumstances this quarter. I know that major technological glitches arose for many of you at various times and you let none of them stand in your way. Best of luck in your future careers and keep me posted on your doings!