Lecture 6-3

We turn now to the linear algebra portion of the course. Although in the course I covered systems of linear equations before vector spaces, I will now review vector spaces first (following the the LADW text). We have seen that every finite-dimensional vector space V (one spanned by finitely many vectors) is such that any two bases of it (any two spanning sets sets which are also linearly independent) have the same number of vectors, called the dimension of V. If the dimension of V is n, then any $m > n$ vectors in V are necessarily linearly *dependent*, while no $m < n$ vectors in V can span V. Any linearly independent subset of V can be enlarged to a basis of V; any spanning set for V (finite or infinite) can be shrunk to a basis of V. Given an $m \times n$ matrix A, the row space of A is the subspace of $\mathbb{R}^n$ spanned by its rows, while the column space (or range) of A is the subspace of $\mathbb{R}^m$ spanned by its columns. Although these are subspaces of different vector spaces, they always have the same dimension, called the rank of A. The nullity of A, defined to be the dimension of the nullspace of A (consisting of all column vectors v with $Av = 0$), has dimension equal to n minus the rank of A; the left nullspace of A, consisting of all *row* vectors w with $wA = 0$, has dimension equal to m minus the rank of A.

These subspaces and their dimensions are most readily computed using row operations on A; such operations, used originally in the course to decide whether a linear system $Ax = b$ is solvable and work out all of its solutions whenever it is solvable, preserve the row space of A while sending its column space to another subspace of $\mathbb{R}^m$ with the same dimension. The rank of A equals the number of pivots in any echelon form of A; it is because these pivots appear symmetrically, with at most one of them in every row and column of A that the row and columns spaces of A have the same dimension. In particular, an $n \times n$ matrix A has linearly independent rows if and only if it is have linearly independent columns, or if and only if its rows span $\mathbb{R}^n$, or if and only if its columns span $\mathbb{R}^n$. All of these conditions are equivalent to the invertibility of A; that is, to the existence of an $n \times n$ matrix $B = A^{-1}$ with $AB = I$; whenever such a matrix B exists we automatically have $BA = I$ also. If A is square and invertible then the unique solution to any linear system $Ax = B$ that makes sense is $A^{-1}b$; in general, we determine whether $Ax = b$ is solvable by first augmenting the matrix A, adding the vector b as its new rightmost column, bringing A to echelon form, applying all row operations to the column b as well as the entries in the columns of A, and then seeing whether any zero rows in the echelon form end in a nonzero in the last column. The system $Ax = b$ is solvable if and only if there are no such rows in the echelon form of the augmented matrix; if so, then columns of this echelon form *without* pivots (not counting the last column) correspond to free variables in the linear system. These variables can take any values; they completely determine the other variables.

There is a single number $\det A$ attached to any $n \times n$ matrix A called its determinant which is nonzero if and only if A is invertible. It is most readily calculated by bringing A (as usual) to echelon form by row operations. If a multiple of one row of A is added to another, then $\det A$ is unchanged; if a row of A is multiplied by a scalar C, then $\det A$ is also multiplied by c; if two rows of A are interchanged then $\det A$ changes by a sign. The same results hold for column operations performed on A and in fact any square matrix A has the same determinant as its transpose A^T. Once A is in echelon form and in particular is upper triangular, then its determinant is the product of its entries along the main diagonal. We also have the twin expansions $\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A^{ij}$ and $\det A = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A^{ij}$ of $\det A$ about the ith row or jth column of A, where A^{ij}, the ij-minor of A, is obtained from A by deleting its ith row and jth column. If B is the transpose of the cofactor matrix of A, so that its ijth entry is $(-1)^{i+j}A^{ji}$, the jith cofactor of A, then $AB = BA = (\det A)I$.

Two $n \times n$ matrices A, B are called similar (or conjugate) if there is an invertible $n \times n$ matrix P with $B = P^{-1}AP$; if so then $B^k = P^{-1}A^kP$ and A^k are also similar for any nonnegative integer k, or for any integer if A and B are invertible. Similar matrices have the same rank, the same determinant, and the same trace (sum of their diagonal entries). I will say more about similar matrices when I review the connections between matrices and linear transformations next time.