Lecture 6-2

Reviewing the material from multivariable integration, we have seen that a real-valued function f defined and continuous on a nice enough subset S of $\mathbb{R}^n$ is integrable over S, where "nice enough" means that if S in enclosed in an n-dimensional rectangle $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and if f is extended to R by decreeing that it has the value 0 at any $x \in R$ with $x \notin S$, then f is continuous on R except on a set of measure 0. Typically the subset S can be described by inequalities, or more precisely by inequalities on each of the the variables in some order, the inequalities depending only on subsequent variables in the order. In this case the measure-zero condition is always satisfied and moreover we can evaluate the integral of f over S by antidifferentiating f with respect to each of its variables in turn in the given order, with limits given by the inequalities. The choice of limits for each variable is dictated by the order in which the variables are integrated (as well as by the subset S), but this order never has any effect on the integrand.

There are a number of important interpretations of multiple integrals; first, the n-volume (volume in n dimensions) of a region S in $\mathbb{R}^n$ equals the n-fold integral of the constant function 1 over S. If a physical object occupies the region S in $\mathbb{R}^n$ with the density at any point $\vec{x} \in S$ being given by the function $\delta(\vec{x})$, then the mass of the object equals the n-fold integral of $\delta(\vec{x})$ over S. The centroid $(\bar{x}_1, \ldots, \bar{x}_n)$ then has $\bar{x}_i = \frac{\int_S x_i \delta(\vec{x})\, d\vec{x}}{\int_S \delta(\vec{x})\, d\vec{x}}$; here the numerator $\int_S x_i \delta(\vec{x})\, d\vec{x}$ is called the ith moment of the object.

The change of variable formula for multiple integration says the following. Suppose that we have a function $g : \mathbb{R}^n \to \mathbb{R}^n$ and subsets S, T of $\mathbb{R}^n$ such that the restriction of g to S defines a one-to-one differentiable function from S to T with a differentiable inverse and write $g(x_1, \dots, x_n) = (u_1, \dots, u_n)$. Given a continuous real-valued function f on T the composite function $f(g)$ is real-valued and continuous on S. Then we have $\int_S f(g)\, du_1 \dots du_n = \int_T f|J|\, dx_1 \dots dx_n$, where J is the Jacobian matrix $\partial(u_1, \dots, u_n)/\partial(x_1, \dots, x_n)$ of the u_i with respect to the x_i and $|J|$ is the absolute value of its determinant. There is an extra hypothesis in the text, namely that $\det J$ have constant sign on T, but in fact this is not necessary, as we can always partition T into subregions on which this determinant has constant sign and then apply the formula to each one. For example, if we want to integrate a continuous function $f(x, y)$ on the union of two circular sectors, defined in polar coordinates via $r \in [0, 1], \theta \in [-\pi/4, \pi/4]$ and $r \in [0, 1], \theta \in [3\pi/4, 5\pi/4]$, then we can instead integrate $|r|f(r \cos\theta, r \sin\theta)$ over the rectangle $r \in [-1, 1], \theta \in [-\pi/4, \pi/4]$; although the determinant r of the Jacobian matrix does not have constant sign on this rectangle, it is the union of two other rectangles on which this determinant does have constant sign. The other thing that might worry you about this example is that the map sending (r, θ) to $(r \cos\theta, r \sin\theta)$ is not in fact one-to-one on the entire rectangle; this objection is answered by the following paragraph.

The formula continues to hold if there are open subsets S', T' of S, T, respectively, g is one-to-one with a differentiable inverse on S', mapping it onto T', and the complements of S' in S and T' in T both have measure 0. In the example of the last paragraph, we had $S = [-1, 1] \times [-\pi/4, \pi/4]$ and T was the union of two circular sectors T_1 and T_2; taking S' to be the union of $(0, 1] \times [-\pi/4, \pi/4]$ and $[-1, 0) \times [-\pi/4, \pi/4]$ and T' to be T minus the origin, we find that the above properties hold for S' and T', so the change of variable formula indeed holds in this example.

The most common changes of variable are given by the map $g(r, \theta) = (r\cos\theta, r\sin\theta)$ on the plane, the map $f(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ on $\mathbb{R}^3$, and the map $g(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, also on $\mathbb{R}^3$; of course the first of these is called the polar coordinate map and the other two are the cylindrical and spherical coordinate maps, respectively. As you saw on the second midterm, however, you should be prepared to work with modifications of these coordinate systems (for example by rescaling the coordinates) and be able to recompute $|J|$ after such modifications.