Lecture 5-8

Continuing from last time, we evaluate a number of integrals by changing to polar coordinates. First, the sphere of radius R centered at the origin may be viewed as the region between the graphs of $z = -\sqrt{R^2 - x^2 - y^2}$ and $z = \sqrt{R^2 - x^2 - y^2}$ over the disk D of radius R centered at the origin in the xy-plane. Hence the volume of the sphere is given by $\int_D 2\sqrt{R^2 - x^2 - y^2}, dx\, dy$, which we rewrite first as an iterated integral and then switch to polar coordinates, obtaining $\int_0^{2\pi} \int_0^R 2r\sqrt{R^2 - r^2})\, dr\, d\theta$; note that here, as in many examples to come, the appearance of the extra factor of r in the integrand makes the integral much easier to evaluate. Then we get $\int_0^{2\pi} -(2/3)(R^2 - r^2)^{3/2}|_0^R\, d\theta = \int_0^{2\pi}(2/3)R^3\, d\theta = (4/3)\pi R^3$, in accordance with the formula we all know and love. A somewhat more involved but not really more difficult calculation leads to the volume of the spherical cap obtained if the sphere of radius R is cut by the plane $z = \sqrt{R^2 - R_0^2} < R$ and only the portion above this plane retained. This portion is the region between the graphs of $z = R_0$ and $z = \sqrt{R^2 - x^2 - y^2}$ lying over the disk D' of radius R_0 in the xy-plane; its volume is given by the integral $\int_0^{2\pi} \int_0^{R_0} r((\sqrt{R^2 - r^2} - \sqrt{R^2 - R_0^2}\, dr\, d\theta = (2\pi/3)(R^3 - (R^2 - R_0^2)^{3/2}) - \pi R_0^2\sqrt{R^2 - R_0^2}$.

Next we do a very classical computation which leads unexpectedly to another improper integral in one variable. Integrating the function $f(x,y) = e^{-x^2-y^2}$ over the disk D_R of radius R, using polar coordinates, we get $\int_0^{2\pi}\int_0^R re^{-r^2}\,dr\,d\theta = \int_0^{2\pi} -(1/2)e^{-r^2}|_{r=0}^{r=R}\,d\theta = \pi(1-e^{-R^2})$; taking the limit as $R \to \infty$ we get π. Now $\lim_{R\to\infty}\int_{-R}^R\int_{-R}^R f(x,y)\,dx\,dy$ turns out to have the same value π; to see this, note that if we integrate f over the region C_R bounded between D_R and the rectangle $S_R = [-R,R]\times[-R,R]$, the integrand is positive and bounded above by e^{-R^2}, while the area of the region of integration is $(4-\pi)R^2$, so that the difference between the integrals of f over S_R and D_R is bounded by $(4-\pi)R^2e^{-R^2}$, which goes to 0 as $R \to \infty$. But now $\int_{S_R} f(x,y)\,dx\,dy = \int_{-R}^R\int_{-R}^R e^{-x^2-y^2}\,dx\,dy = \int_{-R}^R e^{-y^2}(\int_{-R}^R e^{-x^2}\,dx)\,dy = (\int_R^R e^{-x^2}\,dx)^2$. We conclude that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ (as we stated earlier but did not prove then). The function $g(x) = e^{-x^2}$, after a suitable change of the independent variable x, represents the classical normal or Gaussian distribution, important in both probability and statistics; a large number of attributes such as human height, depending on a large number of more or less independent factors, are distributed in this way.

Of course our earlier formula $\int_a^b (1/2)f(\theta)^2\,d\theta$ for the area of the region between the polar graph $r = f(\theta)$ and the rays $\theta = a, \theta = b$. also follows at once from double integration in polar coordinates (of the function 1 over the region in question); we used a special case of this formula to derive the change of variable factor r for integration in polar coordinates in the first place. We conclude our treatment of polar coordinates with yet another classical computation, this time of the volume of a right circular cone with base radius R and height h. The height of this cone varies linearly from h at $(0,0)$ to 0 on the circle of radius R in the xy-plane, so is given in polar coordinates by $(h/R)(R-r)$. Accordingly, the volume of the cone is $\int_0^{2\pi}\int_0^R (rh/R)(R-r)\,dr\,d\theta = \int_0^{2\pi} h(\frac{R^2}{2} - \frac{R^2}{3})\,d\theta = (1/3)\pi R^2 h$, again as we all know and love.

As useful as the polar coordinate system is, it is but one of infinitely many coordinate systems, each well adapted to its own kind of double integrals over planar regions. A simple but very useful change of coordinates (in effect) takes place when we have a region R in the plane and we stretch it by constant factors of $a, b > 0$ in the x and y directions, respectively, replacing all points (x, y) in R by new points (ax, by) to obtain a new region R'. Given an integrand $f(x, y)$ defined on R, we get a new integrand g on R', declaring that its value $g(ax, by)$ at t(ax, by) equals $f(x, y)$ at the old point (x, y). What is the relationship between $\int_R f(x, y)\, dx\, dy$ and $\int_{R'} g(x, y)\, dx\, dy$? This is easy to analyze directly, since a typical subrectangle S appearing in an upper or lower sum for the integral of f over R on which f has the greatest lower and least upper bounds m and M, respectively, corresponds to a subrectangle S' on which g has the same greatest lower and least upper bounds m and M, such that the area of S' is ab times that of S. The same result holds if a or b is allowed to be negative, provided that we replace ab by its absolute value $|ab|$. Accordingly, $\int_{R'} g(x, y)\, dx\, dy = |ab| \int_R f(x, y)\, dx\, dy$. For example, the area of the elliptical disk defined by $(x^2/a^2) + (y^2/b^2) \leq 1$ (with $a, b > 0$) is just ab times the area of the unit disk, defined by $x^2 + y^2 \leq 1$, or πab, because the first region is obtained from the second by multiplying all x-coordinates by a and all y-coordinates by b. In general, we express the relationship between the integrals of f and g above formally as follows: we change the names of the variables in the integral of g to u and v and write $x = u/a, y = v/b$, whence formally $dx\, dy = \frac{1}{|ab|} du\, dv$ (since $dx = \frac{1}{|a|} du, dy = \frac{1}{|b|} dv$; we use absolute value signs here because we are fixing the order of the limits of integration, always going from the smaller limit to the larger one). Hence $\int_R f(x, y)\, dx\, dy = \frac{1}{|ab|} \int_{R'} g(u, v)\, du\, dv$, with a typical point $(u, v) \in R'$ corresponding to $(x, y) \in R$. We will further develop and massively generalize this formalism later.