Lecture 5-7

We have seen improper integrals of functions of one variable; thus $\int_a^\infty f(x)\,dx$ is defined to be $\lim_{b\to\infty}\int_a^b f(x)\,dx$, provided the limit exists. Similarly an improper double integral like $\int_a^\infty\int_b^\infty f(x,y)\,dx\,dy$ is defined to be $\lim_{c,d\to\infty}\int_a^c \int b^d f(x,y)\,dx\,dy$ if the limit exists; in this last limit the parameters c,d approach ∞ independently, so that if $\int_a^\infty\int_b^\infty f(x,y)\,dx\,dy = L$, then given $\epsilon > 0$ there is M such that whenever $c,d > M$ we have $|\int_a^c\int_b^d f(x,y)\,dx\,dy - L| < \epsilon$.

One can change the order of integration in both proper and improper double integrals. Remarkably enough, a number of improper integrals of functions of one variable are easier to compute if they are rewritten as double integrals. For example, recall the integral $\int_0^\infty \sin x/x\,dx$ from the fall. We saw then that this integral converges conditionally that is, it converges only because of cancellation between the positive and negative values of the integrand), but we had no means of actually computing it (though we stated at the time that its value is $\pi/2$). We now show how to do this using double integrals. We begin with the simple calculation that an antiderivative of $e^{-ax}\sin x$ is given by $e^{-ax}(\frac{-1}{a^2+1}\sin x - \frac{1}{a^2+1}\cos x)$, where a is a constant. Evaluating $\int_0^\infty \int_0^\infty e^{-xy}\sin x\,dy\,dx$ we get $\int_0^\infty (\sin x/x)\,dx$. Changing the order we get $\int_0^\infty \int_0^\infty e^{-xy}\sin x\,dx\,dy$, which equals $\int_0^\infty e^{-xy}\frac{-y}{y^2+1}\sin x - \frac{1}{y^2+1})\cos x)|_0^\infty = \int_0^\infty \frac{1}{y^2+1}\,dy = \pi/2$, as claimed. Remarkably enough (and again as claimed previously) the integral $I' = \int_0^\infty (\sin x/x)^2\,dx$ also has the value $\pi/2$. To prove this we look at the integral $I = \int_0^\infty \frac{1-\cos x}{x^2}\,dx$. Using integration by parts (with $f(x) = 1-\cos x, g'(x) = 1/x^2$ we find that $I' = -(1-cosx)\frac{1}{x}|_0^\infty + \int_0^\infty (\sin x)/x\,dx = \pi/2$ (since $(1-\cos x)/x \to 0$ as $x \to 0$, by L'Hopital's Rule). Now $1-\cos x = 2\sin^2(x/2)$; making the substitution $u = x/2, du = (1/2)dx$, we get $I' = I$, as claimed.

It should come as no surprise to learn that the integral $\int_R f(x,y)\,dx\,dy$ can be interpreted as the volume of the solid bounded by the region R in the xy-plane and the graph of $f(x,y)$, counting that portion of the solid above the xy-plane positively and the portion below this plane negatively, as the definition of the integral was chosen with exactly this application in mind. As a consequence we can often exploit symmetry to show that certain double integrals must be 0, since the positive part of the integrand cancels out the negative part. Thus we have $\int_D \frac{\sin xy}{x^2+y^2},dx\,dy = 0$ if D is the disk defined by the inequality $x^2+y^2 \le 1$. Double integrals have physical interpretations as well; for example, if an object with density $\rho(x,y)$ at the point (x,y) occupies the region R in the plane, then the mass M of the object is given by $\int_R \rho(x,y)\,dx\,dy$. The center of mass (or centroid) of the object has coordinates $(\bar{x},\bar{y})$, where $\bar{x} = M_x/M$ and M_x is the *x-moment* $\int_R x\rho(x,y)\,dx\,dy$; similarly $\bar{y} = M_y/M$, where M_y is the y-moment $\int_R y\rho(x,y)\,dx\,dy$. Thus the portion D' of the disk D_a of radius a defined by $x^2+y^2 \le a^2$ lying in the first quadrant and having unit density has mass $(\pi/4)a^2$, this being the area of the disk. The x-moment $M_x = \int_{D'} x\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^a x\sqrt{a^2-x^2}\,dx = -(1/3)(a^2-x^2)^{3/2}|_0^a = a^3/3$ and the x-coordinate $\bar{x}$ of the centroid is $\frac{(1/3)a^3}{(\pi/4)a^2} = 4a/3\pi$. By symmetry $\bar{y} = \bar{x}$, so the centroid is located at $(4a/3\pi, 4a/3\pi)$.

Since we often have occasion in practice to integrate functions over disks or other regions readily describable in polar coordinates rather than rectangles it makes sense to ask whether there is a better way to divide such regions into smaller ones other than subrectangles in order to define the upper and lower sums whose limit is a given integral over the region. More precisely, suppose that our region R of integration is defined by the inequalities $a \leq \theta \leq b, g(\theta) \leq r \leq h(\theta)$ in polar coordinates, where $[a, b] \subset [0, 2\pi]$ and $0 \leq g(\theta \leq h(\theta)$ for $\theta \in [a, b]$. Dividing R into "polar rectangles" R_{ij} specified by inequalities $r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j$, we know that the area A_{ij} of R_{ij} is given by $(1/2)(r_i^2 - r_{i-1}^2)(\theta)j - \theta_{j-1})$, from our earlier formula for the area of a region enclosed by a polar graph. Letting m, M be the respective greatest lower and least upper bounds of the integrand f, now regarded as a function of the polar coordinates r and θ rather than x and y thanks to the change of coordinate formulas $x = r\cos\theta, y = r\sin\theta$, we see that upper and lower bounds for the integral of f over R are given by sums to which the contribution of R_{ij} is MA_{ij} and mA_{ij}, respectively. As these last quantities may be regarded as the integrals of Mr and mr over the intervals $r \in [r_{i-1}, r_i], \theta \in [\theta_{j-1}, \theta_j]$, treating r, θ for this purpose as though they were Cartesian coordinates in their own right. Taking the limits of these upper and lower bounds as the polar rectangles R_{ij} get smaller and smaller, we see that *the integral* $\int_R f(x, y)\, dx\, dy = \int_a^b \int_{g(\theta)}^{h(\theta)} r f(r\cos\theta, r\sin\theta)\, dr\, d\theta$. This is the *change of variable formula for polar coordinates*, which we will explore more fully next time.