Lecture 5-6

Last time we learned that if a real-valued function $f(x,y)$ is integrable over a rectangle $R = [a,b] \times [c,d]$ and if for each fixed $y \in [c,d]$ the function sending $x \in [a,b]$ to $f(x,y)$ is integrable (as a function of one variable), then we can compute the double integral $\int_R f(x,y)\,dx\,dy$ as an iterated integral $\int_c^d \int_a^b f(x,y)\,dx\,dy$; of course the analogous result holds if instead we assume that the function sending y to $f(x,y)$ is integrable for each $x \in [a.b]$, interchanging the order of integration in the iterated integral. In particular, we can compute any double integral by an iterated integral whenever the integrand is continuous.

As mentioned last time it is much too restrictive just to look at functions defined on rectangles; we also want to integrate functions defined on more general regions R. We do this by enclosing R inside a rectangle R', extending the integrand $f(x, y)$ to R' by decreeing that it be 0 off of R, and then integrating it over R'. In practice we don't want to have to choose the rectangle R' explicitly every time. If the region R can be described by a pair of inequalities $a \leq x \leq b, g(x) \leq y \leq h(x)$, so that it consists exactly of those points satisfying these inequalities, then we can set up the iterated integral of f over R by using variable limits, or more precisely as $\int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$. Note that the innermost limits in an iterated integral of this kind are allowed to depend on the outermost variable, but the outermost limits must be constant (so that the final answer is a number). Similarly, if instead the region R is described by the inequalities $c \leq y \leq y, g(y) \leq x \leq h(y)$, then $\int_R f(x, y)\, dx\, dy$ would be evaluated as $\int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy$.

In practice it often happens that the region R of integration can be described in either of the above ways; in that case it is up to us to choose the order of integration so as to make evaluating the integral as easy as possible. For example, given the double integral of $\sin y/y$ over the triangle T in the xy-plane with vertices $(0.0), (0,1)$, and $(1,1)$, it is easiest to describe T by the inequalities $0 \leq y \leq 1, 0 \leq x \leq y$, leading to the iterated integral $\in_0^1 \int_0^y (\sin y/y)\, dx\, dy = \int_0^1 \sin y\, dy = 1 - \cos 1$. Integrating in the other order would give $\int_0^1 \int_x^1 (\sin y/y)\, dy\, dx$, which we cannot evaluate since there is no elementary antiderivative of $\sin y/y$ with respect to y. The example of $g(, x, y) = x$ being integrated over the triangle with vertices $(0, 0.((1,1)$, and $(2,0)$, is somewhat different; here the order of integration makes no difference as far as antidfferentiating the integrand is concerned. Nevertheless, it is still better to integrate with respect to x first, as then the lower and upper limits are given uniformly by y and $2 - y$, respectively, while y runs from 0 to 1, so the integral equals $\int_0^1 \int_y^{2-y} x\, dx,\, dy = \int_0^1 (1/2)(2-y)^2 - y^2)\, dy = \int_0^1 (2-2y)\, dy = 1$. Had we integrated in the other order, we would have been forced to break up the iterated integral into two subintegrals, as y ranges from 0 to x if $0 \leq x \leq 1$, but from 0 to $2 - x$ if $1 \leq x \leq 2$.

By interchanging the order of integration we get a new proof that $f_{xy}(a_0, b_0) = f_{yx}(a_0, b_0)$ whenever the second-order partials f_{xy}, f_{yx} of a function f defined on a rectangle $R = [a, b] \times [c, d]$ are continuous on R and (a_0, b_0) lies in the interior of R. Indeed, if equality failed, we would have to have $f_{xy}(a_0, b_0) > f_{yx}(a_0, b_0)$ or $f_{xy}(a_0, b_0) < f_{yx}(a_0, b_0)$; for definiteness assume the former. By continuity there would be a subrectangle $R' = [a', b'] \times [c', d']$ lying in R such that $f_{xy} > f_{yx}$ on R', whence $\int_{R'} f_{xy}\, dx\, dy > \int_{F'} f_{yx}\, dx\, dy$. Evaluating the first integral with respect to y first and then x and the second integral in the other order, we would get $f(b', d') - f(a', d') - f(b', c') + f(a', c')$ in both cases (by the Fundamental Theorem of Calculus), a contradiction.

We now say a few words about integrability of functions of two variables. We saw in the fall that a function f of one variable is integrable on an interval $[a, b]$ if and only if it is continuous on that interval except on a set of measure 0. This means that, if D is the set of points $x \in [a, b]$ such that f is discontinuous at x and if $\epsilon > 0$ there is a countable collection of intervals $[a_1, b_1], (a_2, b_2], ldots$ such that $D \cup_i [a_i, b_i]$ and $\sum_i (b_i - a_i) < \epsilon$. A similar criterion holds for real-valued functions on the xy-plane; such a function is integrable over a region R if and only if it is continuous at all points of R except those in a subset D and for every $\epsilon > 0$ there is a countable collections of subrectangles whose union contains D and the sum of whose areas is less than ϵ.

As an example, if R is the region between the graph of a continuous function $g = f(x)$ and the interval $[a, b]$ on the x-axis and if f is a real-valued function continuous on R then f is integrable. This is not obvious, since we have seen that in order to define the integral of f over R we must enclose R in a rectangle R' and then extend f to R' by decreeing that it is 0 off of R, thereby making it discontinuous at (typically) all points on the graph of $g(x)$. To see that this graph is a set of measure 0, recall that $g(x)$ is integrable on $[a, b]$ and so given $\epsilon > 0$ there is a partition P of $[a, b]$ such that the upper and lower sums of $g(x)$ attached to this partition differ by less than ϵ. For each subinterval $[x_{i-1}, x_i]$ in this partition, if we construct a rectangle $R_i = [x_{i-1}, x_i] \times [m_i, M_i]$, where m_i, M_i are the respective greatest lower and least upper bounds of $g(x)$ on $[x_{i-1}, x_i]$, then the union of the R_i contains the graph and the sum of the areas of the R_i is less than ϵ, as desired. Thus the additional discontinuities arising in the graph of f when it is extended to R' do not cause a problem for integration.