Lecture 5-5

We turn now to multivariable integration, the final topic for the sequence, beginning with an account of double integrals. These are defined in much the same way as single integrals; they are intended to rigorously define the notion of the volume between the graph of a function $f(x, y)$ and a rectangle $R = [a, b] \times [c, d]$, counting this volume as negative whenever the graph of $f(x, y)$ lies below the xy-plane and positive whenever it lies above this plane. Given $f(x, y)$ and $R = [a, b] \times [c, d]$, we begin with a pair of partitions $P = \{x_0, \ldots, x_n\}, Q = \{y_0, \ldots, y_m\}$ of $[a, b]$ and $[c, d]$, respectively, so that $x_0 = a < \ldots < x_n = b, y_0 = c < \ldots < y_m = d$. For every pair of indices i, j with $1 \leq i \leq n, 1 \leq j \leq m$, let m_{ij}, M_{ij} be the respective greatest lower and least upper bounds of f on the subrectangle $[x_{i-1}, x_1] \times [y_{j-1}, y_j]$. Define the *lower* and *upper* sums $L(f, P, Q), U(f, P, Q)$ by $L(f, P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$ and $U(f, P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$. As with integrals of functions of one variable, one can refine (P, Q) to another pair (P', Q') of partitions, thereby increasing the lower sum and decreasing the upper sum of f relative to the new pair (P', Q'), and then use refinements to show that any lower sum of f (relative to any partition pair) is less than or equal to any upper sum (again relative to any partition pair). Then we say that f is *integrable on* R if the least upper bound of the $L(f, P, Q)$ coincides with the greatest lower bound of the $U(f, P, Q)$ (over all partitions P, Q); if so we denote their common value I by $\int_R f(x, y)\, dx\, dy$. Here I deliberately use just one integral sign rather than two (unlike the text) because I want to emphasize that only a single act of integration is taking place here. We will learn below how to compute such an integral as an *iterated integral*, in which we integrate f first with respect to x and then with respect to y.

We can give a simple example of an integral computed directly from the definition, closely paralleling an analogous example we gave earlier for a function of one variable. Take $[a,b]=[c,d]=[0,1], R=[a,b\times[c,d]$ and $f(x,y)=xy$. Here f is strictly increasing on R in both x and y. Setting P and Q to be the equal partitions $\{0,1/n,\ldots,1\},\{0,1/m,\ldots,1\}$ we have $m_{ij}=\frac{(i-1)(j_1)}{nm}, M_{ij}=\frac{ij}{nm}$, whence $L(f,P,Q)=(1/mn)\sum_{i=1}^n\sum_{j=1}^m\frac{(i-1)(j-1)}{mn}=\frac{m(m-1)n(n-1}{4m^2n^2}$; similarly $U(f,P,Q)=\frac{m(m+1)n(n+1)}{4m^2n^2}$. Here it is quite easy to check that $(1/4)$ is the unique number greater than all lower sums but less than all upper sums, whence $\int_R]xy\,dx\,dy=1/4$. On the other hand, if $g(x,y)=0$ if both $x,y\in\mathbb{Q}$ and $g(x,y)=1$ otherwise, then every lower sum of g over R is 0 and every upper sum is 1, so g is not integrable over R.

Thus integration of a function of two variables, unlike differentiation of such a function, is a straightforward extension of doing the same thing for functions of one variable. On the other hand, double integration is more complicated than single integration in that we also want to integrate functions over more general regions S than rectangles; for example, we would like to compute the area of S by doubly integrating the constant function 1 over it (as it turns out we can indeed do). Given a bounded region S, a natural way to integrate a real-valued function f over S would be to locate a rectangle R containing S, extend the domain of f to R by decreeing that $f(x,y) = 0$ if $(x,y) \in R, (x,y) \notin S$, and then integrate f over R. A difficulty with this approach, however, is that if we do this our integrand f will typically be discontinuous at all boundary points of S when regarded as defined on R. We must therefore be prepared to handle functions with many more points of discontinuity than what we saw in the one-variable case. Fortunately it turns out that the above definition is indeed flexible enough to do this. As an example, consider the function $h(x,y)$ defined on $R = [0,1] \times [0,1]$ by $h(x,y) = 1/q$ if $x, y \in \mathbb{Q}, y = p/q$ in lowest terms, and $p, q \geq 0$, is discontinuous at all points $(x,y) \in R$ with $y \in \mathbb{Q}$, an infinite union of lines. Nevertheless, $\int_R h(x,y)\,dx\,dy$ exists and equals 0. To prove this let $\epsilon > 0$. There are only finitely many $y \in [0,1]$, say $y_1, \ldots, y_m$, with $h(x,y).\epsilon/2$ for any $x \in [0,1]$, as any such y is rational and there are only finitely many possibilities for its numerator and denominator. Choosing a partition Q of $[0,1]$ for the y variable such that the sum of the lengths of the subintervals including one of the y_i is less than $\epsilon/2$ and the trivial partition $P = \{0,1\}$ for the x variable, we find that the upper sum $U(P,Q,h) < \epsilon$, since the contribution to this upper sum from the subrectangles S such that $h(x,y) > \epsilon/2$ for any $(x,y) \in S$ is less than $\epsilon/2$, while we have $h(x,y) \leq \epsilon/2$ for the other subrectangles and their total area is less than 1. Since the upper sums of $h(x,y)$ can therefore get arbitrarily small, while all lower sums are 0, we have $\int_R h(x,y)\,dx\,dy = 0$, as claimed. Thus even functions with many discontinuities indeed have some hope of being integrable over rectangles.

What happens with this particular function $h(x,y)$ is that if we fix $y \notin \mathbb{Q}$ then we have $h(x,y) = 0$ for all $x \in [0,1]$, so that $\int_0^1 h(x,y)\,dx = 0$. If on the other hand we fix $y \in \mathbb{Q}$ with $y = p/q$ in lowest terms, then $h(x,y) = 1/q$ if $x \in \mathbb{Q}$ and $h(x,y) = 0$ otherwise. In this case $\int_0^1 h(x,y)\,dx$ does not exist, since the integrand is discontinuous everywhere. Notice however that if we let $L(y)$ be the *lower* integral of $h(x,y)$ with respect to x and $U(y)$ the upper integral, then we have $L(y) = 0$ for all y, while $U(y) = 1/q$ if $y \in \mathbb{Q}, y = p/q$ in lowest terms and $U(y) = 0$ otherwise. We saw the function $U(y)$ in the fall quarter; it is the standard example of a function discontinuous at all rational points in $[0,1]$ but continuous at the irrational ones. The following term we showed that $\int_0^1 U(y)\,dy = 0$; of course $\int_0^1 L(y)\,dy = 0$ as well. Hence the *iterated integral* $\int_0^1 (\int_0^1 h(x,y)\,dx)\,dy$, in which we integrate first with respect to x, treating y as a constant, and then with respect to y, treating the innermost integral as either a lower or an upper integral, has value 0, like the double integral of the same function. It turns out that the same thing happens for any integrable function $f(x,y)$ on a rectangle $R = [a,b] \times [c,d]$; in particular, if $\int_a^b f(x,y)\,dx$ exists for all y, then we have $\int_R f(x,y)\,dx\,dy = \int_c^d (\int_a^b f(x,y)\,dx)\,dy$. In practice we omit the parentheses around the inner integral, evaluating by convention any iterated integral from the inside out.