Lecture 5-27

We continue our discussion of orthogonality and dot products, this time in a more abstract context. When we were discussing Fourier series, we made the crucial observation that the functions $f_n(x) = \sin nx$ for $n > 0$ and $g_n(x) = \cos nx$ for $n \geq 0$ are such that the integral of the product of any two distinct functions in this list from $-\pi$ to π is 0, while the integral of the square of any function in this list from $-\pi$ to π is π, except for g_0; the integral $\int_{-\pi}^{\pi} g_0^2\,dx = 2\pi$. On a purely formal level, this last statement seems to say that the f_n and g_n form a set of pairwise orthogonal vectors. This is indeed the case! We already know that in fact any continuous function on $[-\pi, \pi]$ can indeed be regarded as a vector, simply because the set C of such functions is a vector space and any element of a vector space can be called a vector. We need to introduce an analogue of the dot product on this vector space to justify regarding these vectors as orthogonal. This analogue should come as no surprise by now, given the integral formulas just mentioned; we decree that if $f, g \in C$, then $f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. (In effect we replace the most naive analogue, namely $\sum_{x\in[-\pi,\pi]} f(x)g(x)$, which does not make sense, by the integral $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, which does). Clearly with this definition $f \cdot g$ is linear in both f and g; furthermore, the crucial property $f \cdot f = 0$ if and only if $f = 0$ also holds, since the integral $\int_a^b g(x)\,dx$ of any nonnegative continuous function $g(x)$ over an interval $[a, b]$ with $a < b$ is 0 if and only if $g = 0$. We express this property by saying the dot product is *positive definite*, using a phrase we have used before in another context.

With the dot product in hand, we can now define the length (or norm) of a function $f \in C$ in the same way as for $\mathbb{R}^n$; we set $||f|| = \sqrt{f \cdot f}$. Then the Cauchy-Schwarz inequality $|f \cdot g| \leq ||f||||g||$ carries over immediately to C, with the same proof as for $\mathbb{R}^n$; we also have the triangle inequality $||f + g|| \leq ||f|| + ||g||$, which we can again prove in the same way as for $\mathbb{R}^n$. In fact, even though the vector space C does not seem at first to have any geometric interpretation, we can in fact do geometry on it, as we can on any vector space equipped with a dot product. In particular, we can speak of the angle between two functions in C, defined in the same way as for $\mathbb{R}^n$. Thus we now can indeed regard the functions f_n, g_n above as pairwise orthogonal vectors in C. Dividing all the f_n, g_n with $n \neq 0$ by $\sqrt{\pi}$ and g_0 by $\sqrt{2\pi}$, we get a family F of orthonormal vectors in C.

Now this family F is not an orthonormal basis of C, as there are clearly functions in C that are not finite linear combinations of the functions in F. What is true however is that functions in the span of F are dense in C; that is, given $g \in C$ and $\epsilon > 0$ there is a finite linear combination f a functions in F (sometimes called a trigonometric polynomial) such that $||f - g|| < \epsilon$. Thus in some sense the vector space C is not as large as it first appears; even though it has uncountable dimension, it admits a countable-dimensional dense subspace. Vector spaces with this last property are called *separable*; unfortunately this meaning has nothing to do with separable differential equations. Thus just as the uncountable set $\mathbb{R}$ admits the countable dense subset $\mathbb{Q}$, so the uncountable-dimensional space C admits a countable dense subset, consisting of all finite linear combinations with rational coefficients of functions in F.

The above family F is by no means the only countable orthonormal set of vectors whose span is dense in C. Another one that looks quite different is the family of *Legendre polynomials*. These are usually regarded as vectors in the space C' of continuous functions on $[-1,1]$ rather than as vectors in C (though of course any polynomial is continuous on all of $\mathbb{R}$). The dot product is naturally enough defined on C' by $f \cdot g = \int_{-1}^{1} f(x)g(x)\,dx$. Legendre polynomials are constructed by starting with the obvious basis $1, x, x^2, \ldots$ of the subspace of polynomials on $[-1,1]$ and then replacing it by an orthogonal (not orthonormal) basis of the same space via the Gram-Schmidt process. The first two polynomials are $P_0 = 1$ and $P+1 = x$; thereafter one has the recurrence relation $(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$, so that P_n has degree n. Instead of $P_n \cdot P_n = 1$ we have $P_n \cdot P_n = \frac{2}{2n+1}$; we can also say that the P_n are normalized by the condition that $P_n(1) = 1$ for all n. There are many beautiful facts about the P_n (which you can look up on Wikipedia and other places); I will content myself with mentioning *Rodrigues's formula*, which asserts that $P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. What makes the span of the P_n dense in C' (or the span of F dense in C) is that there is no nonzero $f \in C'$ with $f \cdot P_n = 0$ for all n (and similarly no nonzero $f \in C$ with $f \cdot g = 0$ for all $g \in F$).