Lecture 5-26

We begin by going over last Friday's midterm. The first problem was unquestionably the toughest. Following the hint, start with a basis $v_1, \ldots, v_m$ of the kernel of f and expand it to a basis $B = v_1, \ldots, v_n$ of V. The added basis vectors $v_{m+1}, \ldots, v_n$ are then such that $w_{m+1} = f(v_{m+1}), \ldots w_n = f(v_n)$ are linearly independent, for otherwise f would send some linear combination of $v_{m+1}, \ldots, v_n$ to 0, whence that combination would have to equal a combination of $v_1, \ldots, v_m$, contradicting linear independence of $v_1, \ldots, v_n$. Consequently $w_{m+1}, \ldots, w_n$ can be extended to a basis $B' = w_{m+1}, \ldots, w_n, w'_1, \ldots, w'_r$ of W. Now the matrix of f relative to the bases B, B' has all zeroes in its first m columns consist of one 1 and all other entries 0, where the 1s start at the first coordinate and and move one coordinate down in every successive column. In particular there is at most one 1 in every row and column and all other entries are 0, as claimed. In the second problem, computing the determinants of the upper left square submatrices of A, we get $2, 1$, and $a + 1 - 2 = a - 1$ (expanding $\det A$ along its last row). Hence A is positive definite if and only if $a > 1$.

In the third problem, we see at once by symmetry that the coordinates $\bar{x}, \bar{y}, \bar{z}$ of the centroid are all equal. Computing $\bar{x}$ by integrating in spherical coordinates, we obtain $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta = \frac{\frac{1}{4}\cdot\frac{\pi}{4}\cdot 1}{\frac{\pi}{6}} = \frac{3}{8}$, whence the centroid is $(\frac{3}{8}, \frac{3}{8}, \frac{3}{8})$, since the volume of the portion of the ball in the first octant is one-eighth of the total volume of $4\pi/3$ of the ball. In the fourth problem we set $x = ar\cos\theta, y = br\sin\theta$, so that the inequalities $0 \le r \le 1, 0 \le \theta \le 2\pi$ describe the region of integration in r, θ coordinates; then the desired mass is $\int_0^{2\pi}\int_0^1 abkr(a^2r^2\cos^2\theta + b^2r^2\sin^2\theta) \, dr \, d\theta = (k/4)(\pi a^2 + \pi b^2)ab$, since the change of variable factor for this coordinate change is rab. In the last problem, let $y = mx + b$ be the linear polynomial. To solve for m and b we start with the inconsistent system $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. The normal equations form the system $\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$, whose solution is $m = 3/2, b = 5/6$. The best least-squares linear fit has equation $y = (3/2)x + (5/6)$.

We now wrap up Chapter 5 in LADW with a further account of orthogonality of vectors in $\mathbb{R}^n$; the discussion extends to $\mathbb{C}^n$ if the dot product is replaced by the Hermitian inner product, but for simplicity we will stick to the real case and the dot product. Given a collection of pairwise orthogonal nonzero vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, the coefficients a_i in any linear combination $v = \sum_{i=1}^m a_i v_i$ are uniquely recovered from v alone, since $a_i = \frac{v \cdot v_i}{v_i \cdot v_i}$. In particular the v_i are necessarily linearly independent in this situation, so that they form a basis for the subspace S of $\mathbb{R}^n$ that they span. If the v_i happen in addition to be unit vectors, so that they form what is called an orthonormal basis of S, then the formula for a_i simplifies even further, to $a_i = v \cdot v_i$. Thus not only is every orthonormal set of vectors automatically a basis for the subspace that it spans, but it is especially easy to write any vector in this space as an explicit combination of vectors in this basis. We actually saw an infinite-dimensional version of this same phenomenon when we were discussing Fourier series. We observed that the integral $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$ for all nonnegative integers n, m, while $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0$ whenever $n \neq m$; moreover, $\int_{-\pi}^{\pi} \sin^2 nx \, dx = \int_{-\pi}^{\pi} \cos^2 nx = \pi$ if $n \neq 0$ but $\int_{-\pi}^{\pi} \cos^2 0x \, dx = 2\pi$. As a formal consequence, if an integrable function $f(x)$ on $[-\pi, \pi]$ is written as the sum $\sum_{n=1}^{\infty} a_n \sin nx + \sum_{m=0}^{\infty} \cos mx$, then we can recover the coefficients a_n, b_n in the sums via $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$ for $n \neq 0$ and $b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos 0x \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$. We can then use the Fourier series for $f(x) = |x|$, for example, which converges to $f(x)$ for $x \in [-\pi, \pi]$, to give another proof that the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Since orthonormal bases are so much easier to compute with than arbitrary ones, it is of interest to know that there is a systematic procedure, called the *Gram-Schmidt process*, for replacing any basis $v_1, \ldots, v_m$ of any subspace S of $\mathbb{R}^n$ by an orthonormal one $u_1, \ldots, u_m$ for the same space. We do this one step at a time. Start by replacing the first vector v_1 by the unit vector $u_1 = v_1/||v_1||$ in its direction. Then replace v_2 by the difference $v_2' = v_2 - (v_2 \cdot u_1)u_1$ of v_2 and its orthogonal projection onto u_1, so that $v_2' \cdot u_1 = 0$; note that v_2 cannot be 0, since v_1, v_2 are independent. Then replace v_2' by the unit vector u_2 in its direction; clearly the span of u_1, u_2 is the same as that of v_1, v_2. Continue by replacing v_3 by the difference $v_3' = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2$ of itself and the sum of its projections onto $u_1 u_2$; then replace v_3' by the unit vector u_3 in its direction, and so on. Then $u_1, \ldots, u_m$ is an orthonormal basis of S, as claimed. For example (see p. 132 of LADW), if $v_1 = (1,1,1), v_2 = (0,1,2), v_3 = (1,0,2)$, then $u_1 = \frac{1}{\sqrt{3}}(1,1,1), u_2 = \frac{1}{\sqrt{2}}(-1,0,1), u_3 = \frac{1}{\sqrt{6}}(1,-2,1)$. Sometimes (as we will see tomorrow) we do not bother to make the new basis orthonormal, but only orthogonal; in that case we can skip the steps involving dividing certain vectors by their lengths.

We now interpret the Gram-Schmidt process in matrix terms. Given a basis $v_1, \ldots, v_n$ of all of $\mathbb{R}^n$, this process shows that we may replace each v_i by a linear combination of itself and various v_j for $j < i$ to produce an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$; moreover, the coefficient of v_i in this combination is positive (more precisely, it is the reciprocal of the length of the vector v_i' defined in the above paragraph). Define an invertible matrix M by decreeing that its i-th row is the vector v_i. By our earlier work on row operations, we know that replacing each v_i by the sum of a positive multiple of itself and a combination of earlier v_j can be implemented by multipliying the matrix M by an invertible lower triangular matrix L with positive entries along its diagonal, so that the i-th row of the resulting matrix $U = LM$ is the i-th vector u_i of the orthonormal basis. But now we have seen that the matrix U, having orthonormal rows, must also have orthonormal columns and in fact must be an orthogonal matrix. We conclude that *any invertible $n \times n$ matrix M is the product LU of a lower triangular matrix L with positive entries along its diagonal and an orthogonal matrix U*. Moreover, *this product decomposition is unique*, for if $x_1 y_1 = x_2 y_2$ with the x_i lower triangular and the y_i orthogonal, then $x_2^{-1} x_1 - y_2 y_1^{-1}$ is both orthogonal and lower triangular with positive entries along the diagonal, whence it is easy to check that $x_2^{-1} x_1 = y_2 y_1^{-1} = I, x_1 = x_2, y_1 = y_2$. This is a special case of something called the *Iwasawa decomposition*, which plays an important role in my field of representation theory.