Lecture 5-21

We now review the calculus we have learned since the first midterm. We have seen that the definition of the double integral of a function of two variables over a rectangle $[a, b] \times [c, d]$ (or of a function of n variables over an n-dimensional rectangle $[a_1, b_1] \times \ldots \times [a_n, b_n]$) is a straightforward extension of the definition in the one-variable case; one partitions the intervals defining the rectangle, uses these partitions to partition the rectangle itself into subrectangles, multiplies the n-dimensional volume of each subrectangle by the greatest lower and least upper bounds of the function on the subrectangle to define the lower and upper sums of the function relative to the partition, and then takes the greatest lower bound of all upper sums to be the integral of the function, provided that this also coincides with the least upper bound of all the lower sums. Whenever f is integrable on a rectangle, we can evaluate its integral as an iterated integral, in which we integrate with respect to each of the variables in turn between the limits of that variable, treating all other variables as constants, finally integrating a function of just one variable between constant limits to get the final result.

This same method applies to integrals over more general regions in n-space, provided that these regions can be defined by limits on each of the variables in turn, in such a way that the variables occur in a certain order and the possibly variable limits for each variable depend only on later variables (so that the limits on the last variable are constants). We then evaluate the integral from the inside out, or from right to left, again recalling that when we integrate with respect to one variable, all other variables are treated as constants. We often have a choice in the order of variables of integration; we must make sure that, once this order has been decided upon, the limits for each variable correctly reflect exactly how large and how small that variable can get once values for the other variables on which these limits depend have been specified. If the same function is integrated in two different orders, the limits of integration will change, depending on the order used, but the integrand itself will not change. Depending on both the integrand and the shape of the region of integration, integration in one order may be much easier than integration in another; it is up to you to work out the best order in each case.

Many regions of integration are such that it is much more natural to divide them into smaller regions other than subrectangles; for example, if one is integrating over a disk, then it is much easier to subdivide the disk into circular sectors, defined by constant limits on the polar coordinates r and θ, than to subdivide the disk into subrectangles; indeed, it is impossible to realize the disk as the union of finitely many subrectangles (overlapping or not). At first we might think we have to reformulate our original definition of the double integral, using circular sectors rather than disks; but fortunately this is not necessary; instead we have a theorem which states that if the region R of integration can be defined by the inequalities $g(\theta) \leq r \leq h(\theta), a \leq \theta \leq b$ (where $0 \leq g(\theta) \leq h(\theta), [a, b[\subset [0, 2\pi]$), then the integral of a continuous function $f(x, y)$ over R may be expressed as $\int_a^b \int_{g(\theta)}^{h(\theta)} r f(r\cos\theta, r\sin\theta)\, dr\, d\theta$; this is the *change of variable formula for polar coordinates*. The extra factor r in the integrand arises because the area of a circular sector defined by the inequalities $a \leq r \leq b, c \leq \theta \leq d$ in polar coordinates is obtained by integrating r with respect to r and θ in turn between the respective limits a, b and c, d. Thus, for example, if one wants to compute the mass of a plate occupying the unit disk in the xy-plane whose density at the point (x, y) is proportional to the cube of its distance to the center $(0, 0)$ of the disk, with constant of proportionality k, then this mass is given by $\int_0^{2\pi} \int_0^1 kr^4\, dr\, d\theta = 2\pi k/5$.

There are two main three-dimensional analogues of polar coordinates, namely cylindrical and spherical coordinates. Cylindrical coordinates are labelled r, θ, z, where r, θ are the polar coordinates of the point projected to the xy-plane and z is the same as in Cartesian coordinates. The change of variable factor in the integration formula is again r, as for polar coordinates. Spherical coordinates are labelled ρ, ϕ, θ; here ρ is distance form the origin, ϕ is latitude, measured down from the north pole, and θ is the same as in cylindrical coordinates; note that ϕ runs from 0 to π only, while θ runs all the way from 0 to 2π. The change of variable factor for spherical coordinates is $\rho^2 \sin \phi$, arising as the integrand in the formula for the volume of a spherical wedge defined by constant limits on ρ, ϕ, and θ.

Finally we considered general coordinate changes. Given a one-to-one differentiable map g with a differentiable inverse from one region $R \subset \mathbb{R}^n$ to another one R', let $|J|$ be the *absolute value* of the Jacobian matrix of g; then the integral of a function f defined on R' equals the integral of the composite function $f(g)$ times $|J|$ over R. We use the absolute value of the determinant of the Jacobian matrix rather than the determinant itself because we interpret it as the ratio of the n-dimensional volumes of two regions in $\mathbb{R}^n$. If g is not one-to-one but the n-dimensional volume of the subset S' of R' consisting of all points y with $f(x) = y$ for at least two $x \in R$ is 0, then the change of variable formula still holds.

The midterm tomorrow will once again be taken on Canvas; I will email it to you by 11:25 or so and you have until 12:25 to turn it in. Good luck.