Lecture 5-19

Given an $n \times n$ matrix A, we showed last time how to define and compute its exponential e^A; we also showed that the law of exponents $e^{sA}e^{tA} = e^{(s+t)A}$ holds for multiples of A (but not in general), We can think of e^{tA} as the t-th power of e^A; note that the set of all matrices e^{tA} as t runs over $\mathbb{R}$ is a subgroup (called a *one-parameter subgroup*) of the group $GL(n,\mathbb{R})$ of invertible $n \times n$ matrices over $\mathbb{R}$ under multiplication. Now in general we cannot define B^t if B is a square matrix and $t \in \mathbb{R}$ is not an integer; even if we are lucky enough to have $B = e^A$ for some A, it is easy to have $e^{A_1} = e^{A_2}$ but $e^{tA_1} \neq e^{tA_2}$. (In fact, we run into the same problem for complex numbers z; even though every complex number has *a* square root, there is no good uniform way to define a square root *function* on the entire complex plane.) If however $A = P^{-1}DP$ and D is diagonal with *positive* real diagonal entries $d_1, \ldots, d_n$, then it is natural to define A^t for any $t \in \mathbb{R}$ as $P^{-1}D^tP$, where D^t is the diagonal matrix with diagonal entries $d_1^t, \ldots, d_n^t$, using the definition we gave in the fall of x^t for any positive x. Moreover, it is not difficult to show that a matrix M commutes with D if and only if it commutes with D^t for any $t \neq 0$, since M turns out to commute with D if and only if its ij-th entry $m_{ij} = 0$ whenever $d_i \neq d_j$ and $d_i^t = d_j^t$ for $t \neq 0$ if and only if $d_i = d_j$. It follows that if $P^{-1}DP = Q^{-1}DQ$, then $P^{-1}D^tP = Q^{-1}D^tQ$, so we can define A^t unambiguously in this setting at $P^{-1}D^tP$.

If $A = P^{-1}DP$ is as above, then by using the Jordan form, one can show for any positive integer n that any matrix B with positive eigenvalues satisfying $B^n = A$ is necessarily diagonalizable and in fact we must have $B = P^{-1}D^{1/n}P$. Thus *diagonalizable matrices with positive real eigenvalues have unique n-th roots with positive eigenvalues for any nonzero integer n.* In particular, *positive definite (symmetric) matrices have unique positive definite n-th roots for every $n > 0$*, just as positive real numbers have unique positive n-th roots for any such n. In fact, even positive *semidefinite* symmetric matrices (whose eigenvalues, by definition, are all nonnegative) have uniquely defined n-th roots for all n. Positive definite matrices have the additional property that their t-th powers are uniquely defined and positive definite for all $t \in \mathbb{R}$; positive semidefinite matrices have the same property if $t \geq 0$. All of this unfortunately breaks down even for symmetric matrices that are not positive semidefinite. For example, the 2×2 matrix $A = -I$ has many square roots among real 2×2 matrices, but no 2×2 *real symmetric* square root, since any such root would have to have eigenvalues $\pm i$, but no real symmetric matrix has imaginary eigenvalues. The existence of unique positive definite square roots for positive semidefinite matrices will play an important role in the lectures next week, where we introduce our final new topic, namely the singular value decomposition.

In a similar way one can show that positive definite matrices P have unique symmetric logarithms A; that is, there is a unique symmetric A with $e^A = P$ (though A need not be positive definite, as A will have a negative eigenvalue whenever P has an eigenvalue less than 1). Not all exponential matrices e^A will have positive eigenvalues, however, since the matrix A can have complex eigenvalues λ and then the eigenvalue e^λ of e^A need not even be real. Any real $n \times n$ matrix A whose negative real eigenvalues occur with even multiplicity turns out to admit a logarithm. So does any matrix $I + B$ where the $n \times n$ matrix B has all entries less than $1/n$ in absolute value. To see this fact, recall the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ for $\ln(1+x)$, which converges for $x \in \mathbb{R}$ with $|x| < 1$. Substituting the matrix B for the variable x and using the previously observed fact that the entries of any power B^k are bounded in absolute value by $n^{k-1}c^k$, where c is an upper bound for the absolute values of the entries of B, we see that this series converges entry by entry to the desired logarithm for $I + B$. More generally, any power series $\sum a_n x^n$ in x with a positive radius of convergence R will also converge entry by entry if a square matrix A whose entries are all less than R/n in absolute value is substituted for x. Such power series other than the exponential one, however, have not found many applications to date.