Lecture 5-15

We conclude our treatment of multivariable integration with some remarks about setting up limits in multiple integrals. We have already seen that we often have a choice in the order of variables of integration; it is important to make sure that the limits of integration reflect this order. Note that the outermost limits must always be constant, so that the final answer is a number. The next to outermost limits are allowed to depend on the outermost variable; the next limits to these are allowed to depend on the two outermost variables, and so on, with the innermost limits allowed to depend on the greatest number of variables. It is often best to set up the limits from left to right even though any iterated integral is always evaluated from right to left. Thus in integrating some function $f(x, y, z)$ over the tetrahedron defined by the inequalities $x, y, z \geq 0, 2x + 3y + 4z \leq 4$, if we integrate in the order $dx\,dy\,dz$, then the outermost limits on z should be 0 and 1, since on this overall region these are the smallest and largest values that z can take. For each fixed z in this range, y runs from 0 to $\frac{4-4z}{3}$; if both z and y are specified, then x runs from 0 to $\frac{4-4z-3y}{2}$. If we integrated in the order $dz\,dy\,dx$ instead, then the x limits would have been 0 and 2, the y limits would have been 0 and $\frac{4-2x}{3}$, and the z limits would have been 0 and $\frac{4-2x-3y}{4}$. Notice also that the integrand $f(x, y, z)$ is not affected by the order of integration.

If the upper or lower limits of integration for a given variable are given by a nonuniform formula then the integral must be broken up into two or more subintegrals which are added to obtain the final result. For example, we have already seen that in integrating a function $f(x,y)$ over the triangle T in the xy-plane with vertices $(0,0),(1,1)$, and $(2,0)$, then if we integrate in the order $dx\,dy$ then we get just one double integral, namely $\int_0^1\int_y^{2-y} f(x,y)\,dx\,dy$; there is a uniform formula for the leftmost and rightmost boundaries of this region of integration. If we integrate the same function $f(x,y)$ in the other order, then the double integral must be broken into two subintegrals, namely $\int_0^1\int_0^x f(x,y)\,dy\,dx+\int_1^2\int_0^{2-x} f(x,y)\,dy\,dx$; there are two formulas for the upper boundary of the region of integration, depending on the value of x.

We give an example of a tricky volume computation that involves subdividing the region of integration in a more subtle way. Suppose that we have three long cylindrical pipes of unit radius placed so that their axes of symmetry are the coordinate axes. What is the volume of their intersection? The pipes are defined by the inequalities $x^2 + y^2 \leq 1, x^2 + z^2 \leq 1, y^2 + z^2 \leq 1$ and we want to find the volume of the set S of points satisfying all three of these inequalities. If $z^2 \geq 1/2$, then the constraints on x and y are $x^2 \leq 1 - z^2, y^2 \leq 1 - z^2$; note that then $x^2 + y^2 \leq 1$ automatically. We get analogous constraints on x and z if $y^2 \geq 1/2$, or on y and z if $x^2 \geq 1/2$. There are no points in S such that two of x^2, y^2, z^2 exceed $1/2$. Finally, if $x^2, y^2, z^2 \leq 1/2$, then automatically $x^2 + y^2, x^2 + z^2, y^2 + z^2 \leq 1$. We have divided our set S into four subsets which overlap only on a set of measure zero in $\mathbb{R}^3$; it only remains to add the volumes of each subset. By symmetry the first three subsets have the same volume. We compute this volume as $2 \int_{\sqrt{2}/2}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dx\, dy\, dz = 2 \int_{\sqrt{2}/2}^{1} 4(1 - z^2)\, dz = \frac{16}{3} - 4\sqrt{2} + 8\frac{\sqrt{2}}{12} = \frac{16}{3} - \frac{10}{3}\sqrt{2}$; here we get the initial factor of 2 since $z^2 \geq 1/2$ if and only if either $z \in [\sqrt{2}/2, 1]$ or $z \in [-1, -\sqrt{2}/2]$ and the integral with $z \in [-1, -\sqrt{2}/2]$ is the same as the one with $z \in [\sqrt{2}/2, 1]$ by symmetry. The total volume of all three is thus $16 - 10\sqrt{2}$. The last subset is the product of three intervals $[-\sqrt{2}/2, \sqrt{2}/2]$ so has volume $2\sqrt{2}$. Adding up, we find that the total volume of S is $16 - 8\sqrt{2}$. A similar but easier computation shows that the volume of the set R of points (x, y, z) in $\mathbb{R}^3$ with $x, y, z \geq 0, x + y, x + z, y + z \leq 1$ is $1/4$.

We close with the last example in §17.9 of the text (p. 928). This asks us to find the volume of the solid T enclosed by the surface with equation $(x^2+y^2+z^2)^2 = 2z(x^2+y^2)$. In spherical coordinates the bounding surface takes the form $\rho = 2\sin^2\phi\cos\phi$. Since z takes only positive values on T, the limits on ϕ are 0 and $\pi/2$, while the limits on ρ are 0 and $2\sin^2\phi\cos\phi$; those on θ are 0 and 2π as usual. Hence the desired volume is $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\sin^2\phi\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = (8/3)(\int_0^{2\pi} d\theta)(\int_0^{\pi/2}(\sin^7\phi\cos\phi - \sin^9\phi\cos\phi)\, d\phi = (2/15)\pi$.

Next week we will return to linear algebra, picking up some additional topics from the Treil notes. We will also treat exponentials of square matrices, which combine linear algebra and differential equations in a very nice way.