Lecture 5-14

We illustrate the general change of variable formula for multiple integrals in a couple of examples. First we compute the area enclosed by the *astroid*, which has the equation $x^{2/3} + y^{2/3} = a^{2/3}$, where a is a positive constant. (This curve is called the astroid because its graph looks like a star.) Although we can solve the equation for y explicitly in terms of x, it is not feasible to antidifferentiate the resulting function, so we use a more roundabout method. Using modified polar coordinates r, θ, with the change of coordinate formulas $x = r\cos^3\theta, y = r\sin^3\theta$ we find that the equation of the astroid reduces to $r = a$. With these coordinates the Jacobian determinant $|\partial(x,y)/\partial(r,\theta)|$ becomes $3r^2(\cos^3\theta\sin^2\theta\cos\theta + \sin^3\theta\cos^2\sin\theta) = 3r^2\cos^2\theta\sin^2\theta = (3/4)r^2\sin^2 2\theta$; the limits on r and θ in the region in $r\theta$-space mapping to the one enclosed by the astroid are $0 \le r \le a, 0 \le \theta \le 2\pi$. Accordingly the desired area is $\int_0^{2\pi}\int_0^a (3/4)r^2\sin^2 2\theta\, dr\, d\theta = (3/8)\pi a^2$; the astroid encloses exactly 3/8 of the area of the circle into which it is inscribed.

Our second example is trickier; it serves to illustrate another useful general fact about Jacobian determinants. This is Example 2 on p. 933 of the text, but we will use a slicker method than the one used there. We are asked to compute the integral $\int_\Omega xy\,dx\,dy$, where Ω is the region is the region in the first quadrant bounded by the curves $x^2-y^2=1, x^2-y^2=4, x^2+y^2=4, x^2+y^2=9$. Here we reverse the usual procedure; instead of realizing Ω as the image of a simpler region under a 1-1 differentiable map from $\mathbb{R}^2$ to $\mathbb{R}^2$, we go the other way and exhibit such a map taking Ω to a simpler region. This map is fairly easy to guess; it sends (x,y) to (x^2+y^2, x^2-y^2), thereby mapping Ω to the rectangle $[4,9]\times[1,4]$. We observe that we can indeed solve uniquely for $x,y\geq 0$ in terms of $u=x^2+y^2$ and $v=x^2-y^2$, but (unlike the text) we do not bother to do this explicitly. Instead we calculate the Jacobian determinant $J=|\partial(u,v)/\partial(x,y)|$, obtaining $|(2x)(-2y)-(2y)(2x)|=8xy$. But now we were supposed to express the integrand xy in terms of the new coordinates u,v and also work out the other Jacobian determinant $J'=|\partial(x,y)/\partial(u,v)|$. Instead of doing this directly we argue that the product of the Jacobian *matrices* $\partial(u,v)/\partial(x,y)$ and $\partial(x,y)/\partial(u,v)$ is $\partial(u,v)/\partial(u,v)=I$,by the chain rule, whence the Jacobian determinants J,J' are multiplicative inverses. Hence the product xyJ' of the original integrand xy and J' is the constant $1/8$ and the desired integral is just $\int_4^9\int_1^4(1/8)\,du\,dv=15/8$. Note that had the integrand been the supposedly simpler constant function 1, the integral would have been harder rather than easier to evaluate.

The reasoning used in the second example helps to illustrate an advanced theorem closely related to the Implicit Function Theorem mentioned in a couple of earlier lectures. Note that in the statement of the general change of variable formula it is crucial that the function g mapping one of the regions R of integration to the other one R' be one-to-one (though actually it is enough if it is one-to-one when restricted to a subset S of R whose complement in R has measure zero and maps it onto a subset S' of R' whose complement also has measure zero; consider the polar coordinate transformation $g : (r, \theta) \to (r \cos\theta, r \sin\theta)$, which maps the nonclosed rectangle $(0, 1] \times [0, 2\pi)$ in a one-to-one fashion onto the unit disk with the origin removed). For example, it would be disastrous to compute the area of the unit circle by integrating $r\,dr\,d\theta$ over say the closed rectangle $[-1/2, 1] \times [0, 3\pi]$, even though this rectangle is mapped onto the unit disk by g. It is therefore of importance to know when a differentiable map from $\mathbb{R}^n$ to itself has a differentiable inverse.

The *Inverse Function Theorem* states that if a continuously differentiable map g sending $(u_1, \ldots, u_n)$ to $(x_1, \ldots, x_n)$ is such that $|\partial(x_1, \ldots, x_n)/\partial(u_1, \ldots, u_n)|$ is nonzero at the point $\vec{a} \in \mathbb{R}^n$, then there are neighborhoods N, N' of $\vec{a}, \vec{g}(\vec{a})$, respectively, such that the restriction of g to N is a one-to-one differentiable function from N onto N' with a differentiable inverse. We express this in words by saying that any differentiable function with a nonzero Jacobian determinant at a point admits a local differentiable inverse. It need not admit a global differentiable inverse, as the example $g(x, y) = (e^x \cos y, e^x \sin y)$ shows; this function has Jacobian determinant $e^{2x} \neq 0$ at (x, y) and yet $g(x, y) = g(x, y + 2k\pi)$ for any integer k. Thus the change of variable formula would not apply (for example) to an integral over the rectangle $[0, 1] \times [0, 3\pi]$ with this function g.