Lecture 5-13

We now introduce the other coordinate system most often used in $\mathbb{R}^3$ (apart from Cartesian and cylindrical coordinates); this is the spherical coordinate system. Its coordinates are labelled ρ, ϕ, and θ, and are defined as follows. The coordinate ρ represents distance from the origin. The coordinate ϕ represents latitude as measured *down from the north pole*, as opposed to up or down from the equator; thus ϕ lies in the interval $[0, \pi]$. Finally, θ is the same as in cylindrical coordinates, in effect corresponding to longitude, so that this coordinate (unlike ϕ) ranges over the entire interval $[0, 2\pi]$. (Note that the notations ϕ and θ are often reversed in the physics literature, so that in physics θ rather than ϕ often represents latitude.) The change of coordinate formulas now read $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$; one way of remembering that $\cos\phi$ rather than $\sin\phi$ goes with z is to recall that z must be allowed to take negative values even though ρ is always positive and ϕ is confined to the first two quadrants. (As with cylindrical coordinates, we do not bother with the reverse formulas for ρ, ϕ, θ in terms of x, y, z, which are uglier and rarely used.)

Now in order to work out the multiplicative factor that should be inserted in an integral when one switches to spherical coordinates, we need to know the volume of a spherical wedge, that is, of a region given by the Cartesian product of three intervals in spherical coordinates. We saw last time that, at least in the special case where the intervals for ρ and ϕ both have left endpoint 0, ϕ has right endpoint α, and θ runs from 0 to 2π, this volume $(4\pi/3)R^3(1-\cos\alpha)$ is given by integrating $\rho^2 \sin\phi$ with respect to ρ, ϕ, and θ over the product of the intervals; since the triple integral over a union of two regions overlapping only on their boundaries is the sum of the integrals over each region, the volume of a wedge is given by integrating $\rho^2 \sin\phi$ over the product of the intervals for ρ, ϕ, and θ in general. Hence *given any continuous real-valued function* $f(x,y,z)$ *defined on a region* R *specified by the inequalities* $a \le \phi \le b, g(\phi) \le \theta \le h(\phi), p(\theta,\phi) \le \rho \le q(\theta,\phi)$, *the triple integral of* f *over* R *is given by* $\int_a^b \int_{g(\phi)}^{h(\phi)} \int_{p(\theta,\phi)}^{q(\theta,\phi)} \rho^2 \sin\phi f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, d\rho\, d\theta\, d\phi$. In Example 1 in the text on p. 927, the mass of a solid ball of radius 1 whose density at any point is proportional to the square of its distance from the origin (say with constant of proportionality k) is computed; it is $k\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^4 \sin\phi\, d\rho\, d\theta\, d\phi = (4/5)k\pi$.

There is an n-dimensional analogue of spherical coordinates in which the coordinate ρ again represents distance from the origin, there are $n-2$ angular coordinates $\phi_1, \ldots, \phi_{n-2}$, each running from 0 to π, and one last angular coordinate θ, running from 0 to 2π. The change of coordinate formulas are $x_1 = \rho \sin\phi_1 \ldots \sin\phi_{n-2} \cos\theta, x_2 = \rho \sin\phi_1 \ldots \sin\phi_{n-2} \cos\theta, x_3 = \rho \sin\phi_1 \ldots \sin\phi_{n-3} \cos\phi_{n-2}, \ldots, x_n = \rho \cos\phi_1$. The change of variable factor turns out to be $\rho^{n-1} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \ldots \sin\phi_{n-2}$. By integrating powers of the sine function we get an alternative way to work out the volume of the n-dimensional unit ball, but the way we did it is much easier.

Now the time has come to give a general account of coordinate changes and change of variable factors in integration. Let $x_1, \ldots, x_n$ be n differentiable functions of $u_1, \ldots, u_n$ such that the function $g : (u_1, \ldots, u_n) \to (x_1, \ldots, x_n)$ is differentiable and one-to-one with a differentiable inverse from a region R in $\mathbb{R}^n$ to another one R'. Let f be a continuous real-valued function on R' and $f(g)$ the composite function on R. Write J for the Jacobian matrix $\partial(x_1, \ldots, x_n/\partial(u_1, \ldots, u_n)$ of the x_i with respect to the u_i and $|J|$ for the absolute value of its determinant. *Then* $\int_R f(g)|J|\,du_1 \ldots du_n = \int_{R'} f\,dx_1 \ldots dx_n$. Formally, we write $dx_1 \ldots dx_n = |J|\,du_1 \ldots du_n$. In particular the n-dimensional volume of R' equals the integral of $|J|$ over R; in fact, this special case implies the general result, since we can break up R into finitely many subregions, each with very small n-volume, then use the map g to induce a similar decomposition of R' and then trap $\int_{R'} f\,dx_1 \ldots dx_n$ between upper and lower sums for $\int_R f|J|\,du_1 \ldots du_n$. (The proof of the volume formula is unfortunately beyond the scope of this course.)

In particular the change of coordinate formulas $x = r\cos\theta, y = r\sin\theta$ for polar coordinates imply that $J = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$, whence $|J| = r$. The computation of J for cylindrical coordinates is essentially the same, the Jacobian matrix for polar coordinates being supplemented by an additional row $(0, 0, 1)$ on the bottom and an additional column $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ on the right, so that its determinant r is the same as for polar coordinates. For spherical coordinates we have $J = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\sin\phi & 0 \end{pmatrix}$. Computation of its determinant is a bit tedious from the definition, but there is a special trick; observe that the columns of J are orthogonal. Whenever this happens, the determinant of the matrix is (up to sign) the product of the lengths of its columns. These lengths are $1, \rho$, and $\rho\sin\phi$, whence $|J| = \rho^2\sin\phi$, as desired.