Lecture 5-12

We give an example of a linear change of variable which quite unexpectedly is applied to evaluate the sum of an important series. Consider the innocent-looking integral $I = \int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy$. (This is actually an improper integral, since the integrand becomes unbounded as $(x,y) \to (1,1)$, but we will soon see that it converges.) On the one hand, recognizing the integrand as the sum of the infinite series $\sum_{n=0}^{\infty}(xy)^n$, we get $I = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$; we have seen that this sum converges but have not yet computed its value. Now, on the other hand, if we make the change of variable $x = u+v, y = u-v$, so that $u = \frac{1}{2}(x+y), v = \frac{1}{2}(x-y)$, we find that the integrand becomes $f(uv) = \frac{1}{1+u^2-v^2}$. The limit of integration are a bit trickier to work out. Clearly u runs from 0 to 1; for fixed $u \in [0,1/2], v$ runs from $-u$ to u, considering the southern and western boundaries of the unit square $[0,1] \times [0,1]$. If instead u is fixed in $[1/2,1]$, then v runs from $u-1$ to $1-u$. Since we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ and the matrix has determinant -2, we finally get $I = 2(\int_0^{1/2} \int_{-u}^{u} f(u,v)\,dv\,du + \int_{1/2}^{1} \int_{u-1}^{1-u} f(u,v)\,dv\,du$. Now it is easy to check that $g(u,v) = \frac{1}{\sqrt{1-u^2}} \arctan \frac{v}{\sqrt{1-u^2}}$ is a v-antiderivative of $f(u,v)$, whence $I = 2(\int_0^{1/2} g(u,v)|_{v=-u}^{v=u}\,du + \int_{1/2}^{1} g(u,v)|_{v=u-1}^{v=1-u})\,du$. We now make the substitution $u = \sin\theta, du = \cos\theta\,d\theta$ in the first integral, replacing it by $\int_0^{\pi/6} 2\theta\,d\theta = \pi^2/36$. In the second integral, we substitute $u = \cos\theta, du = -\sin\theta\,d\theta$ and use the half-angle formula $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$, replacing this integral by $\int_0^{\pi/3} \theta\,d\theta = \pi^2/18$. Adding the two integrals and multiplying this sum by 2, we get $I = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, a famous result of Euler in the late eighteenth century. It is remarkable, by the way, that the sum $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ is known for all integers $k \geq 1$ but the single sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is not known (though it is known to be irrational). If one could somehow evaluate the integral $\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz$ then that would give the sum; but no one has figured out how to do this in over 200 years. Do it as an exercise (just kidding).

As you might imagine there are many useful changes of variable that are nonlinear, each of them giving rise to a change of variable integral formula; indeed, we have already seen polar coordinates as an example in two dimensions. The cylindrical coordinate system in three dimensions is the closest analogue there to polar coordinates; in it we have three coordinates r, θ, z, such that z in cylindrical coordinates is the same as z in Cartesian coordinates, while r, θ are determined by x, y exactly as for polar coordinates. Thus the change of coordinate formulas are $x = r\cos\theta, y = r\sin\theta, z = z$. (The reverse change of coordinate formulas, giving r, θ, z in terms of x, y, z are much more awkward and are rarely used.) Now the volume of a cylindrical wedge, defined by the inequalities $a \leq r \leq b$ (with a, b both positive), $c \leq \theta \leq d$ (with $[c, d] \subset [0, 2\pi]$) and finally $e \leq z \leq f$ is easily seen to be $(1/2)(b^2 - a^2)(d - c)(f - e)$ (this cylindrical wedge is just the Cartesian product of a polar wedge and a closed interval), which in turn is the integral $\int_e^f \int_c^d \int_a^b r\, dr\, d\theta\, dz$. Accordingly, by a simple adaptation of the corresponding argument for polar coordinates, we get the following result: *given a continuous real valued-function* $f(x, y, z)$ *defined on a region* R *specified in cylindrical coordinates by the inequalities* $a \leq \theta \leq b, g(\theta) \leq r \leq h(\theta), p(r, \theta) \leq z \leq q(r, \theta)$, *the triple integral of* f *over* R *is given by* $\int_a^b \int_{g(\theta)}^{h(\theta)} \int_{p(r,\theta)}^{q(r,\theta)} rf((r\cos\theta, r\sin\theta, z)dz\, d\theta\, dr$.

We give a couple of examples. First, suppose we have a solid occupying a right circular cylinder of base radius R and height h, such that the density of the solid at any point is proportional to its height above the xy-plane. What is the mass of the solid? Writing k for the proportionality constant, this mass M is given by the integral $\int_0^{2\pi}\int_0^R\int_0^h krz\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^R (1/2)krh^2\,dr\,d\theta = 2\pi(1/4)kR^2h^2 = \pi kR^2h^2/2$. Second, we ask for the volume V of the "ice cream cone" bounded between the upper hemisphere $z = \sqrt{R^2 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}\tan\alpha$. This volume consists of the region between the upper hemisphere and the xy-plane lying over the disk D of radius $R\cos\alpha$ in the xy-plane centered at the origin with the cylinder with base D and height $R\cos\alpha$ removed and replaced by an inverted cone with vertex at the origin. Hence $V = (\int_0^{2\pi}\int_0^{R\cos\alpha}\int_0^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta) - \pi R^3\cos^2\alpha\sin\alpha + (1/3)\pi R^3\cos^2\alpha\sin\alpha$, which works out to be $(2\pi/3)R^3(1-\cos\alpha)$. When we introduce spherical coordinates, we will see that this last region is a wedge in spherical coordinates, given as the Cartesian product of intervals in the spherical coordinates ρ, ϕ, and θ. We will use this computation to derive the change of variable factor for spherical coordinates.