Lecture 5-11

We now consider integrals of functions of more than two variables. The basic idea is a straightforward generalization of what we have done for two variables: given a real-valued function $f(x_{,1}, \ldots, x_n)$ defined on an n-dimensional rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, we partition each of the intervals $[a_i, b_i]$, use these partitions to divide R into subrectangles, multiply the n-dimensional volume of each of these subrectangles S (the product of the lengths of the subintervals of $[a_i, b_i]$ defining them) by the greatest lower bound of f on S, and add the results to define the lower sum of f relative to the partition; then similarly define the upper sum of f relative to the same partition, replacing greatest lower bounds by least upper ones. As before, all lower sums are bounded above by all upper sums. Then f is integrable over R if and only if the greatest lower bound of all the upper sums coincides with the least upper bound of the lower sums, in which case their common value is called the integral of f over R and denoted $\int_R f\,dx_1 \ldots dx_n$. We can compute this via an iterated integral, integrating with respect to each of the variables in any order, using the appropriate interval of integration for each. Equivalently, if the last coordinate x_n of all points in R ranges over the closed interval $[a, b]$ and if we can work out the integral $I(x)$ of $f(x_1, \ldots, x_n)$ over the cross-section of R consisting of all points with $x_n = x$, for all $x \in [a, b]$, then $\int_a^b I(x)\,dx$ will be the integral of $f(x_1, \ldots, x_n)$ over all of R.

In particular, the n-dimensional volume of any region R in $\mathbb{R}^n$ is (by definition) the integral of the constant function 1 over this region, or rather the integral of the extension of this function to an n-dimensional rectangle R' containing R over R', where the extended function is defined to be 0 off of R. Given any region $R \subset \mathbb{R}^n$, let the new region R' be obtained from R by replacing every $(x_1, \dots, x_n) \in R$ by $(a_1x_1, \dots, a_nx_n)$, where the a_i are nonzero constants. Introduce new variables $u_1 = a_1x_1, \dots, u_n = a_nx_n$. Given an integrable real-valued function f on R, let g be the induced function on R', defined via $g(u_1, \dots, u_n) = f(x_1, \dots, x_n)$. Then g is integrable over R' and $\int_R' g \, du_1 \dots du_n = |a_1 \dots a_n| \int_R f \, dx_1 \dots dx_n$. As in the case of two variables, we formally write $du_1 \dots du_n = |a_1 \dots a_n| \, dx_1 \dots dx_n$, as before using the absolute value signs because we are now always putting the smaller of our two limits of integration for each variable on the bottom.

A standard region in n-space arising often in applications is the *unit n-simplex* S_n, defined by the inequalities $x_i \geq 0, \sum x_i = 1$. We will work out a formula for the volume V_n of S_n, proving it by induction on n. The last coordinates of all points in S_n run over the unit interval $[0,1]$; those points of S_n having last coordinate a fixed $x \in [0,1]$ are such that their first $n-1$ coordinates are those of the rescaling of S_{n-1} with all coordinates multiplied by $1-x$. Accordingly, $V_n = \int_0^1 (1-x)^{n-1} V_{n-1}\, dx = V_{n-1}/n$, whence $V_n = 1/n!$ for all n (since it is easy to compute directly that $V_1 = 1$). We can work out the volume U_n of the n-dimensional unit ball $B_n = \{\vec{x} \in \mathbb{R}^n : ||\vec{x}|| = 1\}$ by a similar but more elaborate computation using polar coordinates. Here there are two base cases; we compute directly that $U_1 = 2, U_2 = \pi$. For fixed values of the last *two* coordinates of a point in B_n corresponding to the polar coordinates r, θ, we find that the remaining $n-2$ coordinates are those of a point in the rescaling of B_{n-2} by $\sqrt{1-r^2}$ in each coordinate. Accordingly, we have $U_n = \int_0^{2\pi} \int_0^1 r(\sqrt{1-r^2})^{n-2} U_{n-2}\, dr\, d\theta = (2\pi/n) U_{n-2}$. For even $n = 2m$ we deduce the elegant formula $U_{2m} = \pi^m/m!$ For odd $n = 2m+1$ we have the less elegant formula $U_{2m+1} = \frac{2^{m+1}\pi^m}{1\cdot 3 \cdots 2m+1}$. In particular we have $U_n \to 0$ as $n \to \infty$; this is not surprising when one bears in mind that the constraint $\sum x_i^2 = 1$ becomes more and more restrictive as the number n of variables gets large. (Similarly, of course, the volume V_n of S_n also goes to 0 as $n \to \infty$.)

We now want to investigate linear changes of variable that are more complicated than rescalings. Specifically, we fix an invertible $n \times n$ matrix A and define new variables $u_1, \ldots, u_n$ by decreeing that $\begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$. Given a region R in $\mathbb{R}^n$, replace each $(x_1, \ldots, x_n) \in R$ by $(u_1, \ldots, u_n)$, thereby obtaining a new region R'. Given an integrable function f on R, we define a new function g on R' by setting $g(u_1, \ldots, u_n) = f(x_1, \ldots, x_n)$; this makes sense because the map f sending $(x_1, \ldots, x_n)$ to $(u_1, \ldots, u_n)$ is one-to-one and onto from R to R'. We want to relate $I = \int_R f\, dx_1 \ldots dx_n$ to $J = \int_{R'} g\, du_1 \ldots du_n$. As you might guess by now, the beautiful relationship is that $J = |\det A| I$, the absolute value of the determinant of A times I. To prove this it suffices by the argument used in the rescaling case to show that *any subrectangle $S \subset \mathbb{R}^n$ is such that the n-volume of $f(S)$ is $|\det A|$ times the volume of S*. By rescaling S we may assume that it is the unit cube $U_n = [0,1] \times \ldots \times [0,1]$. Now we know that an invertible matrix A is the product of elementary matrices E. If we can prove this result for the linear map g corresponding to any such matrix E, then it follows by composition of linear maps and the product rule for determinants that it holds for f. But now this is clear. There are three kinds of elementary matrices, each obtained from the identity matrix either by interchanging two rows of the identity matrix I, or multiplying a row of I by a nonzero scalar, or adding a multiple of one row of I to another. In the first case U_n is sent to itself by g and $|\det E| = 1$; the second case is a rescaling that we have already considered; and in the third case $g(U_n)$ is the region between two graphs of functions with constant difference one on the unit cube U_{n-1} in one lower dimension, whence again $g(U_n)$ has n-volume equal to $\det E = 1$, as desired.