Lecture 4-9

Last time we introduced the notion of linear independence of a set of vectors in a vector space and showed that a linearly independent subset of $\mathbb{R}^n$ can have at most n vectors in it. We also showed that any subset of $\mathbb{R}^n$ spanning (or generating) all of $\mathbb{R}^n$ has at least n vectors in it; finally we showed that a linearly independent subset of n vectors in $\mathbb{R}^n$ necessarily spans all of $\mathbb{R}^n$. The same argument shows that conversely any set of n vectors in $\mathbb{R}^n$ spanning $\mathbb{R}^n$ is necessarily independent. Indeed, if we as usual form a matrix M whose columns are the given vectors and bring it to another matrix N in echelon form, then there must be a pivot in every row of N for every system $NX = B$ or $MX = C$ to have a solution, whence as before there is a pivot in every column of N as well and the only solution X to either $NX = \vec{0}$ or $MX = \vec{0}$ is $X = \vec{0}$, whence the columns of M are indeed independent.

A set of vectors in $\mathbb{R}^n$, or more generally in any vector space V, that both spans V and is linearly independent is called a *basis* of V. Since no subset of $\mathbb{R}^n$ spanning it can have fewer than n vectors and no independent subset of $\mathbb{R}^n$ can have more than n vectors, it follows that every basis of $\mathbb{R}^n$ has exactly n vectors. We also saw that any independent subset of $\mathbb{R}^n$ with n vectors, or snay spanning subseet of $\mathbb{R}^n$ with n vectors, is automatically a basis of $\mathbb{R}^n$. We express this situation by saying that $\mathbb{R}^n$ has *dimension* n; this should surely agree with your intuition. More generally, we say that a vector space V is *finite-dimensional* if it has a finite basis; if so any two bases have the same number of elements and we call this number the dimension of V.

Any vector space V generated by a finitely many vectors v_i is finite-dimensional. To see this, note first that it is obvious if the v_i are independent. If they are not, let $\sum a_i v_i = 0$ be a dependence relation with not all a_i equal to 0. Letting i be the largest index with $a_i \neq 0$, we can solve for v_i in terms of $v_1, \ldots, v_{i-1} : v_i = (-1/a_i)\sum_{j=1}^{i-1} a_j v_j$. Now in any linear combination $\sum_{j=1}^{n} b_j v_j$ we can replace the term $b_i v_i$ in it by $(-b_i/a_i)\sum_{j=1}^{i-1} a_i v_i$, obtaining thereby a combination of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ which equals the original combination. The upshot is that $v_1, \ldots, v_n$ *can be replaced by a proper subset of vectors with the same span*. Iterating this procedure, we eventually replace $v_1, \ldots, v_n$ by a (possibly) smaller set of vectors that is independent but still spans V. Thus any generating set for a vector space V can be shrunk to a basis of V (in fact this holds even if V has no finite basis). Now we can generalize a previous result about $\mathbb{R}^n$: *given any set of vectors $v_1, \ldots, v_n$ generating a vector space V and an independent subset $w_1, \ldots, w_m$ of V, we must have $n \geq m$.* Indeed, applying the above recipe for shrinking a generating set to a basis to the ordered set $w_1, v_1, \ldots, v_n$ we find that this set is not independent, since some combination of the v_i equals w_1. Eliminating the first vector in this list that is a linear combination of the preceding ones, we find that that vector cannot be w_1, since w_1 by itself is also linearly independent. Hence we eliminate some of the v_i to get an independent set of vectors $w_1, v_j, \ldots$ with the same span. Now introduce the vector w_2 in the second position, to make a new list $w_1, w_2, v_j, \ldots$, and repeat the above procedure; we do not eliminate w_2 since w_1, w_2 is independent. Continue in this way; the upshot is that we eliminate at least one v_i every time we introduce a w_j, and never eliminate a w_j, so that indeed the number n of available v_i must exceed or equal the number m of the w_j.

Thus a generating subset of a vector space V must always have at least as many vectors as an independent subset of V. In particular, any two bases of V, since they both generate V and are independent, must have the same number of vectors; we call this number the dimension of V. (More generally, it can be shown that even if V is infinite-dimensional, there is a bijection between any two bases of it.) Returning for a moment to differential equations, we can now describe the solution space to a linear homogeneous nth-order differential equation more precisely: *it is a vector space of dimension* n, and any independent set of n solutions $y_1, \ldots, y_n$ to it is such that the general solution is $\sum c_i y_i$, where the c_i are arbitrary constants. We can find a basis of the solution space by fixing a point t_0 in the domain of all the coefficients in the equation and solving the n initial-value problems given by the equation and the conditions $y^{(i)}(t_0) = 1, y^{(j)}(t_0) = 0$ for $j \neq i$ as the indices i, j run from 0 to $n - 1$. In particular the functions $1, t, \ldots, t^{n-1}$ form a basis of the solution space to the equation $y^{(n)} = 0$.