Lecture 4-8

We continue now to parse what we meant last time when we claimed that any subspace of $\mathbb{R}^n$ looks "just like" $\mathbb{R}^m$ for some $m \leq n$. We have seen that, given a vector space V generated by a finite subset $\{v_1, \ldots, v_m\}$ we get an obvious onto map from $\mathbb{R}^m$ to V sending a tuple $(a_1, \ldots, a_m)$ to the combination $\sum a_i v_i$. In order to say that V really looks like $\mathbb{R}^m$ in this case, however, we would like to know that this map is one-to-one as well as onto, so that every $v \in V$ takes the form $\sum a_i v_i$ for a *unique* choice of $a_i \in \mathbb{R}$. Accordingly, we say that vectors $v_1, \ldots, v_m$ in V are *linearly independent*, or just *independent*, if no two distinct combinations $\sum a_i v_i, \sum b_i v_i$ (i.e. no two such combinations with $a_i \neq b_i$ for some i give the same vector. Equivalently, since $\sum (a_i - b_i) v_i = \sum a_i v_i - \sum b_i v_i$, the vectors $v_1, \ldots, v_m$ are linearly independent if and only if the only combination $\sum a_i v_i$ which equals $0 \in V$ is the zero combination (with $a_i = 0$ for all i). A set of vectors is *dependent* if it is no independent. If S is an arbitrary subset of V then S is called linearly independent if and only if every finite subset of S is linearly independent.

The linear independence of a set of vectors is (by far) the most useful when it appears in conjunction with generation. We call a finite set B of vectors in a vector space V a *basis* of V if it both generates V and is linearly independent, so that every vector in V is a *unique* linear combination of elements in B. It should be clear from this definition that if a vector space V admits a basis of m elements, then it really does look exactly like $\mathbb{R}^m$: in fact there is an explicit bijection from $\mathbb{R}^m$ to V in this case. We will say more about this bijection in a more general context later. For now we give some examples. It is clear that the *standard basis* $\vec{e}_i$ of $\mathbb{R}^n$ really is a basis, where $\vec{e}_i$ is the ith unit coordinate vector, having 1 as its ith coordinate and 0 as the other coordinates. Outside of $\mathbb{R}^n$, the powers $1, x, \ldots, x^n$ form a basis for the vector space of polynomials in one variable x of degree at most n (together with the 0 polynomial, which by convention does not have a degree). This vector space can also be described as the solution space to the differential equation $y^{(n+1)} = 0$. The set of *all* powers $1, x, x^2, \ldots$ of x is a basis for the vector space of all polynomials in x. Similarly, the set of *matrix units* e_{ij}, having a 1 as the ij-th entry and 0 as the other entries, forms a basis for the space $M_{m,n}$ of $m \times n$ matrices if the indices i, j respectively run from 1 to m and from 1 to n.

With the notions of generation, linear independence, and basis in hand, we can now return to linear systems. Rather than just looking at a single linear system $MX = B$, we can fix an $m \times n$ matrix M and ask for which column vectors B this system is consistent. It is not difficult to decide that a typical product MV can be rewritten as $\sum_{i=1}^{n} v_i M_i$, where v_i is the ith coordinate in the column vector V and M_i is the ith column of the matrix M, regarded as a vector in $\mathbb{R}^n$. Thus *the set of vectors B such that $MX = B$ is solvable, or consistent, is exactly the span of the set of columns of M*. We call this space the *column space*, or *column span*, of M. How does linear independence fit into this picture? The columns of M are a linearly independent set of vectors if and only if the homogeneous linear system $MX = \vec{0}$ has only the $\vec{0}$ solution. From our previous work we know that *this happens if and only if the echelon from of M has a pivot in every column*, so that there are no free variables in the homogeneous system $MX = \vec{0}$.

Now that we have related spans and independence to systems of linear equations we can deduce a very important and interesting fact. *For any $m > n$ any set of m vectors in*

$\mathbb{R}^n$ *is linearly dependent*. This holds because if we set up a matrix M whose columns are the vectors in question and bring it to echelon form, there can be at most one pivot per row, whence there cannot be a pivot in every column. Furthermore, for any $m < n$, *no set of m vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$*. Indeed, if we set up a matrix M whose columns are the given vectors, its echelon form N must have a row of zeroes, whence there is $B \in \mathbb{R}^n$ such that the system $NX = B$ has no solution (if the ith row of N is $\vec{0}$, just choose a B with nonzero ith coordinate B_i. Running the operations required to produce N from M backwards, we see that there is $C \in \mathbb{R}^n$ such that $MX = C$ has no solution, so that the column space of M is not all of $\mathbb{R}^n$, as claimed. Putting the two settings of generation and linear independence together, we also deduce that *any independent set of n vectors in $\mathbb{R}^n$ generates $\mathbb{R}^n$*. Given n independent vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^n$ we again form the matrix M whose ith column is $\vec{v}_i$. Then this matrix has a pivot in every row and column when brought to echelon form; in fact there is a pivot in the iith entry of the echelon form N for all i between 1 and n. Thus any system $NX = B$ is solvable, whence so is any system $MX = C$.