Lecture 4-3

We close out Chapter 16 by looking at the last section, where we show how to reconstruct a differentiable real-valued function of two variables from its gradient. We have already seen that a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}^2$, regarded as an ordered pair $(P(x,y),Q(x,y))$ of differentiable real-valued functions on $\mathbb{R}^2$ (and often called a *vector field* on $\mathbb{R}^2$), may or may not be the gradient ∇f of a real-valued function f (sometimes called a *scalar field*). If such an f exists for (P,Q), it is called a *scalar potential* for this pair. Recall that we have seen scalar potentials before (though we did not call them by this name): given a differential equation $P\,dx+Q\,dy=0$, if the pair (P,Q) admits a scalar potential f, so that the equation is exact, then the general solution to the equation is $f(x,y)=c$ for some constant c.

We have also seen that if (P,Q) has a scalar potential f, then we must have $P_y=Q_x$, assuming that both of these functions are continuous, for then both of these functions equal $f_{xy}=f_{yx}$. Hence in particular the vector field (x^2,x) has no scalar potential, and indeed if we try to solve the equations $f_x=x^2, f_y=x$, then we find that we must have $f=(x^3/3)+g(y)$ for some function real-valued function g; but no function of the form $(x^3/3)+g(y)$ can have y-partial equal to x. What about the converse? Does every differentiable pair (P,Q) with $P_y=Q_x$ admit a scalar potential f? This turns out to be a rather subtle question. The answer is yes for functions P,Q with continuous partials defined on a rectangle $R=[a,b]\times[c,d]$. Here we can construct the potential f directly. Choose a point (x_0,y_0) in the interior of R and set $f(x,y)=\int_{x_0}^{x}P(u,y_0)\,du+\int_{y+0}^{y}Q(x,v)\,dv$. Then $f_y=\frac{\partial}{\partial y}(\int_{y_0}^{y}Q(x,v)\,dv)=Q(x,y)$, by the Fundamental Theorem of Calculus, since the first integral does not depend on y and the second integral has y as its upper limit. Next $f_x=\frac{\partial}{\partial x}(\int_{x_0}^{x}P(u,y_0)\,du)+\frac{\partial}{\partial x}(\int_{y_0}^{y}Q(x,v)\,dv=P(x,y_0)+\frac{\partial}{\partial x}(\int_{y_0}^{y}Q(x,v)\,dv)$. Now since the operations of differentiation with respect to x and y commute whenever all partials are continuous, it is plausible that the same is true of differentiation with respect to x and integration with respect to y. Taking this for granted and using the hypothesis that $P_y=Q_x$, we replace the second term above by $\int_{y_0}^{y}\frac{\partial P}{\partial y}(x,v)\,dv=P(x,y)-P(x,y_0)$ and the sum equals $P(x,y)$, as desired (see pp. 859-60 of the text). Thus we have a necessary and sufficient condition for continuously differentiable vector fields to be gradients whenever the vector fields are defined on rectangles.

Problems however can arise for vector fields that are undefined at one or more points. The classical example is the field $(P,Q)=(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2})$. Here one can readily check that $P_y=Q_x$. On integrating P with respect to x one obtains $f(x,y)=\arctan(y/x)$, whose y-partial is indeed equal to Q. Are we good here? Unfortunately we are not: the field (P,Q) is undefined only at the origin $(0,0)$, but our function $f(x,y)$ is undefined on the entire y-axis. Thus we do not really have $\nabla f(x,y)=(P(x,y),Q(x,y))$ at all points where $(P(x,y),Q(x,y))$ is defined. The function $f(x,y)$ may ring a bell here: it is the formula usually given for the polar angle coordinate θ in terms of the Cartesian coordinates x and y. This is not quite accurate, however, since the arctangent function is usually taken to have range $(-\pi/2,\pi/2)$, while the angular coordinate θ takes values all the way from 0 to 2π. What is going on here is while we could *define* θ unambiguously for all $(x,y)\neq(0,0)$ by specifying that $0\leq\theta(x,y)<2\pi$, it would not then be *continuous* on the positive x-

axis; no matter how we fuss and fiddle with the definition of θ on the xy-plane, we cannot get rid of its fundamental discontinuity on an entire curve of points including the origin (though we have considerable freedom in moving this curve around). There is no problem with discontinuity at $(0, 0)$, since P and Q are not defined there, but discontinuity at other points is incompatible with having a gradient equal to (P, Q) at such points. The upshot is that (P, Q) is not a gradient after all. Some of you may go on to see (or have already seen) line integrals of vector fields over parametrized curves in the plane (this is covered in Math 324, the sequel to the first-year 12x sequence). It is a fundamental fact that the line integral of any gradient (field) over a closed curve is 0; but the linear integral of (P, Q) over the unit circle traced once counterclockwise turns out to be 2π; in general, over any closed curve not passing though the origin, the line integral of this field turns out to be an integer multiple of 2π.

Strangely enough it turns out that dimension three behaves better than dimension two in this respect. Any suitably differentiable vector field $(P(x, y, z), Q(x, y, z), R(x, y, z))$ with $P_y = Q_x, P_z = R_x, Q_z = R_y$ defined on a rectangular box $[a, b] \times [c, d] \times [e, f]$ turns out to be a gradient: we have $\nabla f = (P, Q, R)$ for a suitable f, and the same holds if (P, Q, R) fails to be defined at one point of a rectangular box. Only if (P, Q, R) is undefined on an entire curve of points can a difficulty analogous to that of the last paragraph arise.