Lecture 4-29

On Monday we learned that the eigenvalues of any $n \times n$ matrix A are exactly the roots of its characteristic polynomial p, which has degree n. Now a fundamental property of polynomials of degree n with real or complex coefficients is that they have n complex roots, counted with multiplicity. Thus in particular any real (or complex) $n \times n$ matrix A has at least one eigenvalue λ, which however might be complex even if A is real; of course if λ is complex we must expect that the coefficients of a corresponding eigenvector will be complex as well. Such a vector can be found by solving the homogeneous system $(A - \lambda I)v = 0$ as usual, working with complex numbers throughout.

It turns out however that a large class of matrices including the symmetric ones all turn out to have *real* eigenvalues and are diagonalizable to boot. Call an $n \times n$ *complex* matrix A *Hermitian* if $\bar{A}^T = A$; here $\bar{A}^T$ is computed by taking A^T as usual and then replacing every entry by its complex conjugate. Thus any real symmetric matrix is Hermitian; more generally a Hermitian matrix A always has real entries along its main diagonal, while a typical off-diagonal entry a_{ij} satisfies $a_{ij} = \bar{a}_{ji}$. Now for any $v \in \mathbb{C}^n$, say with coordinates $(v_1 + iw_1, \ldots, v_n + iw_n)$ with $v_i, w_i \in \mathbb{R}$, the dot product $v \cdot \bar{v}$ of v and its conjugate $\bar{v}$ is the sum $\sum_{i=1}^n (v_i^2 + w_i^2)$, which is 0 if $v = 0$ and is a positive real number otherwise. Moreover we have $\overline{AB}^T = \bar{B}^T \bar{A}^T$ for any complex matrices A, B, just as $(AB)^T = B^T A^T$. Hence if $v \in \mathbb{C}^n$ is an eigenvector of a Hermitian matrix A with eigenvalue λ, then $\overline{\bar{v}^T A v} = \bar{v}^T \bar{A}^T v = \bar{\lambda} v \cdot \bar{v} = \lambda v \cdot \bar{v}$, forcing $\lambda \in \mathbb{R}$, since $v \cdot \bar{v} \neq 0$. Hence *the eigenvalues of a Hermitian matrix are real*. But even more than this is true. If v is an eigenvector of the Hermitian matrix A with eigenvalue λ and if $w \in \mathbb{C}^n$ is such that $\bar{v} \cdot w = v \cdot \bar{w} = 0$, then $\overline{\bar{v}^T A w}^T = \bar{w}^T A v = \lambda \bar{w} \cdot v = 0$, forcing $\bar{v}^T \cdot Aw = 0$, whence *multiplication by A sends the subspace S of $\mathbb{C}^n$ consisting of all vectors orthogonal to $\bar{v}$ to itself.* Now this subspace S cannot contain v, since $\bar{v} \cdot v > 0$. As the solution space to the single homogeneous equation $\bar{v} \cdot x = 0$, this subspace has dimension $n - 1$, whence a basis of it combined with v gives a basis of $\mathbb{C}^n$. Hence A must have an eigenvector in S (multiplication by A is a (complex) linear transformation from S to itself and so must have an eigenvector). The eigenvalue of this eigenvector, being an eigenvalue of A, must also be real.

More generally, given any subspace W of $\mathbb{C}^n$, say of dimension m, the subspace U of all $x \in \mathbb{C}^n$ with $\bar{w} \cdot x = 0$ for all $w \in W$ (called the *orthogonal complement* of W) has dimension $n - m$, being the solution set to a system of m homogeneous equations (since $\bar{w} \cdot x = 0$ for all $w \in W$ if and only if $\bar{b}_i \cdot x$ for all b_i running over a basis of W. Moreover, no $w \in W$ with $w \neq 0$ lies in U, for otherwise $\bar{w} \cdot w = 0$, a contradiction. It follows that the union of bases for U and for W is a basis of $\mathbb{C}^n$. Thus, given an $n \times n$ Hermitian matrix A, we can start with an eigenvector v_1 of A, say with eigenvalue $\lambda_1 \in \mathbb{R}$, pass to the orthogonal complement C of the subspace spanned by v_1, locate a second eigenvector v_2 of A with v_1, v_2 independent, then pass to the orthogonal complement C' of the span of v_1 and v_2 in $\mathbb{C}^n$, and so on; in the end we see that *there is a basis of $\mathbb{C}^n$ consisting of A-eigenvectors, each with real eigenvalue.* Moreover, the construction of this basis shows that if v and w are two vectors in it, then $\bar{v} \cdot w = v \cdot \bar{w} = 0$.

In particular, if A is real and symmetric, then the above analysis applies and shows that A has real eigenvalues; moreover, since the entries of A are real, its eigenvectors may be taken to have real coordinates as well. In this case all of the conjugations occurring in the above paragraph disappear and we conclude that *given any real symmetric matrix A, the ambient vector space $\mathbb{R}^n$ admits an orthogonal basis of A-eigenvectors* (that is, a basis of A-eigenvectors such that any two basis elements are orthogonal). If we form a matrix U whose columns are n orthogonal eigenvectors for A, then $U^{-1}AU$ is diagonal with real entries. But now we can go a step further: by dividing each of the columns in U by its length, we can arrange for every column in U to be a unit vector, as well as for any two columns of U to be orthogonal. The consequence of this extra property is that $UU^T = I$, since a typical entry of UU^T is the dot product of two columns of U, and so is 1 if the two columns are the same and 0 otherwise. Hence $U^T = U^{-1}$, so that $U^T AU = U^{-1}AU = D$, a diagonal matrix. Matrices U with this last property are called *orthogonal*; they correspond (relative to the standard basis) to orthogonal linear transformations, which (by definition) preserve lengths of vectors and angles between vectors in $\mathbb{R}^n$. We will have more to say about such transformations later.